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Value Bounds and Best Response Violations in Discriminatory Share Auctions.*

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Abstract

This paper analyzes a discriminatory share auction in which bidders submit non-increasing step functions with a bounded number of steps, the type space consists of private non-increasing marginal valuation functions, and the number of participants is random. I show that the interim utility can be written as a simple functional of the distribution of the allocated quantity. This allows me to derive equilibrium existence and to give a characterization of the equilibrium bid schedules in terms of the individual bidders' optimality conditions. The characterization facilitates the formulation of bounds on the estimates of marginal valuations between the submitted quantity points and permits a simple estimator of the fraction of best response violations among the submitted bids. Proofs of concept for the bounds and the estimator are given by using a novel data set from meat import quota auctions in Switzerland.

Keywords: Discriminatory Share Auctions, Random Participation, Estimation

JEL-Classifications: D44, C57

1 Introduction

In a share auction bidders compete for the allocation of a divisible good by submitting demand schedules for the shares of the good. Such auctions are a popular means to sell, for example, treasury bonds, electricity, environmental permits, or import quotas. Recently, the empirical interest in such share auctions has surged, and the main interest lies on the most common variant of the share auction in which the bidders' demand schedules must be of the form of a vector of price-quantity points with an upper bound on the permitted number of such points (cf. Kastl, 2011; Hortaçsu and Kastl, 2012; Cassola et al., 2013).

At the center of the research on such step function share auctions is the observation that the bidders' optimality conditions with respect to the submitted quantities yield a map from the demand schedules and the equilibrium distribution of the received quantity to the marginal valuation at the submitted quantity points. This map allows the marginal valuation at the submitted quantity points to be identified from the data: The demand schedules can be directly observed, and the estimator of the equilibrium distribution of the received quantity can be approximated with a simple resampling procedure that returns the empirical distribution

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of a series of realized quantities which are constructed by repeatedly drawing from the set of submitted demand schedules (Kastl, 2011).

Two caveats apply to this unarguably very elegant approach: First, the resampling procedure makes assumptions on the type space (including, for example, bidder independence and ex ante bidder symmetry both within and across the auctions covered by the data) that are neither directly falsifiable with the data nor indirectly with the optimality conditions used. Needless to say, starting from invalid assumptions on the type space potentially leads to erroneous estimation. Second, the characterization of the equilibrium demand schedules using the optimality conditions with respect to the submitted quantities only identifies the marginal valuations at the submitted quantity points, so that the marginal valuation between the submitted quantity points remains unidentified. Even though the assumption of non-increasing marginal valuations allows to derive a lower bound on the whole support and an upper bound from the second step onward by using the point estimates at the submitted quantities, an upper bound on the first step remains unfeasible. Consequently, it is only feasible to draw inference from the bidding data to a lower bound on realized rents in the auction, but not so to an upper bound. Furthermore, the bounds are unlikely to be very tight when the number of submitted steps is few.

This paper revisits and extends the underlying theoretical model for the case of a discriminatory payment rule (developed in Kastl (2012), applied in Hortaçsu and Kastl (2012) to Canadian treasury auctions), and addresses these two caveats: First, it describes a procedure to bound the bidders' marginal valuations between the submitted quantity points over the whole support, and thus is the first to formulate an upper bound on the marginal valuations for the first step. Second, it presents an estimator for the share of price-quantity points that violate best response behavior for a given set of assumptions on the type space made in the resampling procedure. The estimator can thus be used as a selection criterion between different type space assumptions that is based on the degree of bidder irrationality required so that the model is consistent with the observed data.

The discriminatory share auction in which bidders must submit non-increasing step-functions studied here is a variant of the discriminatory share auction studied in Wilson (1979) where bidders are allowed to submit any non-decreasing bid schedule. I assume that the size of the good is normalized to one, the value of the good to the seller is zero, and bidders are characterized by a non-increasing marginal valuation function on the shares of the good. The distribution of valuations is atomless, and the valuations are private and possibly dependent. The auctioneer has zero valuation for the good, sets the clearing price equal to the highest price such that aggregate demand is greater than unity, considers all quantity-price pairs with prices weakly above the clearing price and resolves potential ties with a pre-announced tie-breaking rule. The payment rule is discriminatory, that is, each bidder pays the area under his bid schedule up to the received share.

The model differs from that in Kastl (2012) in three respects. First it allows for (exogenous) stochastic participation: besides drawing the marginal valuations for the bidders, nature decides which subset of bidders is active in the auction. This is equivalent to assuming that with some probability each bidder has a valuation of zero for the good, and hence does not participate. Fluctuating numbers of bidders over time, albeit a prevalent feature of most series of share auctions conducted, have not yet been incorporated either in the theoretical model nor in the estimation. Second, the tie-breaking rules considered here are more general than that in Kastl (2012) who only considers the pro-rata-on-the-margin rule. The tie-breaking rules that I consider allocate the tying bidders with a quantity lying in the set of shares at which the clearing price is submitted. That is, the tie-breaking rules also include, for example, random rationing rules as analyzed in McAdams (2008) for multi-unit auctions.

Third, while Kastl (2012) considers finite-dimensional types that determine the marginal valuation function of the bidders, the type space considered here is that of all uniformly bounded non-increasing functions on $[0,1]$. Such an infinite type space seems to be most natural given that the bidders are characterized by marginal valuation functions.

In Section 2, I show existence of an equilibrium in distributional strategies (Proposition 1). To do so, I first consider an auction with a discrete action space for which existence is established with the arguments of Milgrom and Weber (1985). Using results developed in Reny (1999) and Reny (2011), I then show that, by considering equilibria along a sequence of auctions with discrete action spaces that becomes dense in the continuous action space, I can construct a converging sequence of ϵ-equilibria whose limit is an equilibrium of the auction with the continuous action space. This approach is an alternative to that employed in Kastl (2012) who first constructs a sequence of equilibria using a sequence of auctions with ever finer grids on the action spaces that are such that no ties can occur and then shows that the limit of this equilibrium sequence is an equilibrium in the auction with the continuous action space.

The insights from the proof of the existence result together with Lemma 1 – establishing that the interim utility in the share auction is the continuous analogue to the interim utility of a bidder in the multi-unit auction as derived in McAdams (2008) – allow me to derive the equilibrium characterization in Proposition 2. The characterization makes use of the optimality conditions both with respect to the submitted quantities and with respect to the submitted prices by a bidder, and thus goes beyond the characterization given in Kastl (2012) who only derives the optimality conditions with respect to the submitted quantities. The additional structure on the equilibrium bids gained from the optimality conditions with respect to the prices allows me both to construct the bounds on the marginal valuations between the submitted quantities as well as to formulate an estimator for the share of price-quantity pairs that violate the best response.

Section 3 derives pointwise bounds on the marginal valuations. To do so, I resort to the optimality conditions with respect to prices. Technically, the optimality conditions with respect to prices are special cases of homogeneous Fredholm equations of the first kind which have an infinite number of solutions (cf. e.g. Jerri, 1999). I show how the assumption of decreasing marginal valuations can be fruitfully employed to construct two maps that return for any hypothetical upper bound on the valuations a greatest lower bound on the set of valuations that satisfy the optimality conditions, and conversely, for any hypothetical lower bound on the valuations a least upper bound on the set of valuations satisfying the optimality conditions. It is straightforward to establish that there is a partial order such that these two maps have a set of fixed points which is a complete lattice and which contains the least upper bound and the greatest lower bound of the set of valuations satisfying the optimality conditions with respect to prices. This implies that the least fixed point of the maps constitutes an upper and a lower bound on the set of such valuations. I present a simple fixed point iteration algorithm to find this least fixed point for which the initial conditions – here: rough initial guesses of the bounds – can be obtained from the data.

Section 4 first presents the estimator of the equilibrium distribution of the received quantities, which is then, in a second step, used for the estimation of the share of submitted price-quantity pairs that violate best response behavior. The estimator of the share of best response violations makes use of the observation that, while the optimality conditions with respect to quantities uniquely determine the marginal valuation at any submitted quantity for a given bid schedule and a given equilibrium distribution of the received quantity, the optimality conditions with respect to prices can be used to construct upper and lower bounds on the valuations at the submitted quantity points for a given bid schedule and a given equilibrium distribution of the received quantity. Together, this gives a necessary condition for a price-quantity pair to be consistent with best response behavior. Crucially, this condition depends on the bid schedule and the equilibrium distribution of the received quantity alone, and can hence be tested with the data. The share of best response violations thus estimated then gives us a picture of how much irrationality in the bidders' behavior has to be assumed if the assumptions on the type space underlying the resampling procedure to approximate the winning probability estimator are taken to be valid. The implementation of this resampling procedure, together with the implementation of the fixed point iteration to obtain the bounds, is presented in Section 5.

Section 6 finally presents the respective proofs of concept for the bounds and the estimator of best response violations using bidding data from the monthly Swiss import quota auctions for high quality beef. The aim of Section 6 is twofold: First, I show the result of the algorithm to bound valuations under the assumption that bidders play best response behavior for individual bidders, and then compare estimates of lower bounds on the total ex post rent in the auctions obtained with the algorithm to those obtained with more naive bounds. Second, I emphasize the importance of having a correct notion of the type space when running the resampling procedure: I compare different type space variants, and show that the fraction of best response violations varies greatly across the scenarios. All proofs are found in the appendix.

2 The Model

2.1 Types, Actions, Strategies

A perfectly divisible good is sold by auction. The size of the good is normalized to one, and the value of the good to the seller is zero. The number of potential bidders is $n \geq 2$, and the potential bidders are indexed by $i = 1, ..., n$.

Before the auction starts, nature assigns each bidder bidder i with a non-increasing function $v_i : [0,1] \to [0, \bar{v}]$, $\bar{v} < \infty$, returning the marginal valuation $v_i(q)$ that bidder i attaches to share $q \in [0,1]$ of the good. I call V the space of possible valuations v_i and assume it is equipped with metric d_v induced by the supremum norm. Valuations are private information, but they need not be independent. The commonly known distribution of valuation profiles $v = (v_1, ..., v_n) \in V^n$ is described by a probability measure η on the Borel subsets of V^n with a marginal distribution η_i on V for each bidder i.

(A1) The measure η is absolutely continuous with respect to the product of its marginals, i.e. with respect to $\eta_1 \times ... \times \eta_n$.

(A2) For all bidders $i = 1, ..., n$, it holds that, if $X \subset V$ satisfies $\eta_i(X) > 0$, then there are $X', X'' \subset X$ with $\eta_i(X'), \eta_i(X'') > 0$ where $\forall f \in X'$ and $\forall g \in X''$ it holds that $f(q) > g(q)$, $\forall q \in [0,1]$.

Assumption (A1) is satisfied, for example, if the individual bidders' valuations are independent (cf. Milgrom and Weber, 1985, for a discussion). Assumption (A2) states that any set of valuations with strictly positive measure contains two sets of strictly positive measure where all elements of one set are greater than all elements of the other set under the pointwise partial order. I write $v_{-i} = (v_1, ..., v_{i-1}, v_{i+1}, ..., v_n)$ for the elements of type profile v other than bidder i's type, and denote by $\eta_{-i}(v_{-i}|v_i)$ the distribution of opponent profiles v_{-i} given the realized valuation of bidder i is v_i.

I model stochastic participation by assuming that besides the realization of valuations nature decides for each bidder i whether that bidder is allowed to be active in the auction or not. I capture this by the variable a_i, where a value $a_i = 1$ indicates that bidder i is active and a value $a_i = 0$ indicates that bidder i is inactive. I assume that a_i and v_i are independent for every bidder i. As with the valuations v_i, nature's draw of a_i is private information, but participation need not be independent across bidders. The commonly known probability that participation profile $a = (a_1, ..., a_n) \in \{0,1\}^n$ realizes is given by $g(a)$, where it holds that $\sum_{a \in \{0,1\}^n} g(a) = 1$. I define $g_{-i}(a_{-i}|a_i)$ as the probability that the opponent participation profile $a_{-i} \in \{0,1\}^{N-1}$ realizes given participation of bidder i is a_i.

The active bidders i submit $k \geq 1$ price-quantity pairs $(p_i^j, q_i^j) \in [0, \overline{p}] \times [0,1]$, $j = 1, ..., k$, where $0 < \overline{p} < \infty$ denotes the maximal price bid to be submitted. The number k is fixed and common knowledge among all potential bidders. A feasible action of active bidder i is a k-tuple b_i of price-quantity pairs,

$$b_i = \left\{ \left(p_i^1, q_i^1\right), \left(p_i^2, q_i^2\right), ..., \left(p_i^k, q_i^k\right) \right\} \in \left[[0, \overline{p}] \times [0,1] \right]^k,$$

where the price-quantity pairs (p_i^j, q_i^j) satisfy $\overline{p} \geq p_i^j \geq p_i^{j+1} \geq 0$, and $0 \leq q_i^j \leq q_i^{j+1} \leq 1$, $j = 1, ..., k$, $p_i^{k+1} = 0$, $q_i^{k+1} = 1$. This gives us the set B of feasible actions for the active bidders.

Inactive bidders, on the other hand, are not allowed to submit a demand schedule. Rather, nature submits for every inactive bidder i a bid b_i from the set B_0 of zero bids given by

$$B_0 = \{b_i \in B : p_i^j = 0, \ j = 1, ..., k\}.$$

Under the auction rules that I will consider such a zero bid never results in a strictly positive quantity allocated and neither affects the winning probability of the active bidders. The bids of all active and inactive bidders are taken together in the vector $b \equiv (b_1, ..., b_n) \in B^n$ of feasible bids that I call a bid profile.

I consider distributional strategies (Milgrom and Weber, 1985): A feasible strategy for bidder i conditional on being active in the auction consists in a probability measure μ_i over the product of bidder i's action space and type space, that is, over $B \times V$, where the marginal distribution of the type space is η_i. In other words, for any $X \subset V$ we have $\mu_i(B \times X) = \eta_i(X)$. The set of all such probability measures on $B \times V$ is denoted by $\mathscr{M}$, and individual strategies are collected in the strategy profile

$$\mu = (\mu_1, ..., \mu_n) \in \mathscr{M}^n.$$

I write μ_{-i} for those elements in the strategy profile μ other than the strategy of bidder i, that is, $\mu_{-i} = (\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_n)$, and $\mu_i(.|v_i)$ for bidder i's distribution over B conditional on valuation v_i.

2.2 Allocation Rules

It will be convenient to use the price-quantity tuples $b_i \in B$ to define left-continuous step-functions $\beta_{b_i} : [0,1] \to [0, \overline{p}]$ as

$$\beta_{b_i}(q) = \sum_{j=1}^{k} p_i^j \cdot \mathbb{1}\{q \in (q_i^{j-1}, q_i^j]\},$$

where $\mathbb{1}\{.\}$ denotes the indicator function and I assume $q_i^0 = 0$ and $\beta_{b_i}(0) = p_i^1$. Let

$$\beta_{b_i}^{-1}(p) = \max\left\{ q \in [0,1] : \beta_{b_i}(q) \geq p \right\}$$

be the quantity demanded by player i at price p with the maximum of the empty set taken to be 0 by convention.[1]

For a realized bid profile $b \in B^n$ that has at least one bidder submitting at least one strictly positive price-quantity pair, the auctioneer chooses the price $p^c > 0$ – henceforth called the clearing price – such that the market clears or, if there is no such price, equal to the lowest strictly positive price bid submitted, that is,

$$p^c = \max\left\{\max\left\{p \in [0,\overline{p}] : \sum_{i=1}^{n} \beta_{b_i}^{-1}(p) \geq 1\right\}, \min\left\{p \in \{p_i^j\}_{i\in\{1,\ldots,n\},j\in\{1,\ldots,k\}} : p > 0\right\}\right\},$$

where, again, the maximum of the empty set taken to be 0. The clearing price p^c determines the allocation of the good: if the total demand at p^c is weakly smaller than one, then all demand at p^c is served. If, on the other hand, total demand at p^c is strictly greater than one, then at least one bidder will be rationed. There are two possibilities for this to happen: First, there might be a single bidder i submitting a bid $p_i^j = p^c$ for some $j \in \{1,\ldots,k\}$ with total demand exceeding one, and second, there might be more than one bidder i submitting $p_i^j = p^c$ for some $j \in \{1,\ldots,k\}$ (where the index j might differ for the bidders). In either case, the allocation rule gives priority to high bids, that is, first the price-quantity pairs with prices strictly above the clearing price are served. Then, in the first case, the bidder having submitted $p_i^j = p^c$ for some $j \in \{1,\ldots,k\}$ is allocated the residual supply at p^c. In the second case – which I will henceforth call a tie – rationing among the tying bidders occurs according to some commonly known rationing rule.

Formally, rationing rules are represented as follows: For a given bid profile $b \in B^n$, the rationing rule together with the allocation rule discussed in the paragraph above induces a distribution $H^b : [0,1]^n \to [0,1]$ of the received quantities $q_i^c \in [0,1]$, where $\sum_{i\in N} q_i^c = 1$ holds with certainty. Every bidder i is faced with a marginal distribution H_i^b of the received quantity $q_i^c \in [0,1]$ which is assumed to be measurable in b for every $q \in [0,1]$. While the price-quantity pairs above the clearing price p_c are allocated with probability one, the received quantity $q_i^c \in [0,1]$ for the tying bidders lies in the interval at which the submitted price is equal to the clearing price with probability one:

(A3) For a given clearing price p^c, any bidder $i \in \{1,\ldots,n\}$ and for bids $j \in \{1,\ldots,k\}$ it holds

$$\begin{aligned} p^c < p_i^j &\Rightarrow \lim_{q \uparrow q_i^j} H_i^b(q) = 0 \\ p^c = p_i^j &\Rightarrow H_i^b(q_i^j) - \lim_{q \uparrow q_i^{j-1}} H_i^b(q) = 1 \\ p^c > p_i^j &\Rightarrow H_i^b(q_i^{j-1}) = 1. \end{aligned}$$

As will be established below, it follows from (A3) together with (A2) that whenever there are ties with positive probability, there are tying bidders that strictly prefer to avoid the tie, implying that ties happen in equilibrium with a probability of zero. This in turn allows me to show equilibrium existence and is also important for the equilibrium characterization.

Rationing rules satisfying (A3) include the pro-rata-on-the-margin rationing rule analyzed by Kastl (2012) and the share auction analogue to the random rationing rule employed in McAdams (2003) for multi-unit auctions, but excludes the pro-rata allocation rule discussed in Kremer and Nyborg (2004). This is without loss of practical generality as the pro-rata-on-the-margin rationing rule is the only rule used in practice (Kastl, 2012). In Appendix A, I

[1] Because I have assumed $\beta_{b_i}(q)$ to be right-continuous, the max-operator is properly defined. By this assumption, the max-operator is properly defined in the following construction of S_i^{-1} and p^c as well.

discuss the pro-rata-on-the-margin rationing rule and the random rationing rule in some more detail.

2.3 Interim Utility

The payoff u_i that bidder i of type $v_i \in V$ receives when bids $b = (b_i, b_{-i}) \in B^n$ are submitted is given by

$$u_i(b_i, b_{-i}, v_i) = \int_0^1 \int_0^{q_i^c} \left[v_i(q) - \beta_{b_i}(q)\right] \mathrm{d}q \mathrm{d}H_i^b(q_i^c). \tag{1}$$

I define $\mu_i^{a_i=1}(b_i|v_i) = \mu_i(b_i|v_i)$ and let $\mu_i^{a_i=0}(b_i|v_i)$ put all mass on the set B^0 of zero bids. Because the received quantity for any bidder is unaffected by inactive bidders submitting zero bids, the function

$$W_i^j(q|b_i, v_i, \mu_{-i}) = \sum_{a_{-i} \in \{0,1\}^{n-1}} \left[1 - \int_{V^{n-1}} \int_{B^{n-1}} H_i^b(q) \mathrm{d}\mu_1^{a_1}(b_1|v_1) \ldots \mathrm{d}\mu_{i-1}^{a_{i-1}}(b_{i-1}|v_{i-1}) \times \right.$$
$$\left. \mathrm{d}\mu_{i+1}^{a_{i+1}}(b_{i+1}|v_{i+1}) \ldots \mathrm{d}\mu_n^{a_n}(b_n|v_n) \mathrm{d}\eta_{-i}(v_{-i}|v_i)\right] g(a_{-i}|a_i = 1)$$

returns the (decreasing and right-continuous) probability that the received quantity q^c for active bidder i with valuation v_i strictly exceeds $q \in [q_i^{j-1}, q_i^j)$ given the submitted demand schedule is b_i, and the opponent strategy profile is μ_{-i}. This allows me to derive a simple representation of the interim utility $\Pi_i(b_i, v_i, \mu_{-i})$ of bidder i having valuation v_i when submitting a bid schedule b_i and the opponents following the strategies in strategy profile μ_{-i}.

Lemma 1. *The interim utility for bidder $i \in N$ with valuation $v_i \in V$ when the opponent strategy profile is $\mu_{-i} \in \mathscr{M}^{n-1}$ is given by*

$$\Pi_i(b_i, v_i, \mu_{-i}) = \sum_{j=1}^{k} \int_{q^{j-1}}^{q^j} \left[v_i(q) - p_i^j\right] W_i^j(q|b_i, v_i, \mu_{-i}) \mathrm{d}q. \tag{2}$$

Observe that Lemma 1 does not require assumption (A3). The assumption will be used, however, both for equilibrium existence and the characterization of the equilibrium, to which we turn now.

2.4 Equilibrium

I look for equilibria in distributional strategies that I simply call equilibrium in the following.

Definition 1 (Equilibrium)**.** *A strategy profile μ is called an equilibrium iff*

$$\mu_i \in \arg\max_{\tilde{\mu}_i \in \mathscr{M}} \int_{V \times B} \Pi_i(b_i, v_i, \mu_{-i}) \mathrm{d}\tilde{\mu}_i$$

holds for all bidders $i = 1, \ldots, n$.

Combining the existence argument in Milgrom and Weber (1985) for a discrete action space and the arguments developed in Reny (1999) establishing conditions for the convergence of sequences of ϵ-equilibria to an equilibrium, I arrive at the following existence result.

Proposition 1. *Assume (A1)–(A2) and a rationing rule complying with (A3). An equilibrium exists.*

I now turn to the characterization of the equilibrium bid-schedules. For a submitted opponent bid profile $b_{-i} \in B^{n-1}$, let

$$S_i^{b_{-i}}(p) = 1 - \sum_{j \in \{1,\dots,n\}\setminus i} \beta_{b_j}^{-1}(p)$$

be the residual supply function faced by bidder i at a price $p > 0$, and let the set

$$B_{p,q} = \left\{x \in B^{n-1} : S_i^x(p) \leq q\right\}$$

be the set of opponent bid profiles b_{-i} such that the residual supply $S_i^{b_{-i}}(p)$ at $p \in (0,\bar{p}]$ is weakly below $q \in [0,1]$. Given a strategy profile $\mu \in \mathscr{M}^n$, I write

$$F_{S_i(p)}^{\mu,a_{-i}}(q|v_i) = \int_{V^{n-1}} \int_{B_{p,q}} \mathrm{d}\mu_1^{a_1}(b_1|v_1)\dots\mathrm{d}\mu_{i-1}^{a_{i-1}}(b_{i-1}|v_{i-1})\times$$
$$\mathrm{d}\mu_{i+1}^{a_{i+1}}(b_{i+1}|v_{i+1})\dots\mathrm{d}\mu_n^{a_n}(b_n|v_n)\mathrm{d}\eta_{-i}(v_{-i}|v_i)$$

for the cumulative distribution function of the stochastic residual supply $S_i(p) \in (-\infty,1]$ faced by bidder i with valuation v_i at price $p > 0$ when the realized participation profile is $a_{-i} \in \{0,1\}^{n-1}$ and the other bidders play according to their strategies in the strategy profile $\mu \in \mathscr{M}^n$. I let μ^* denote an equilibrium, and define

$$W_i^*(p,q|v_i) = 1 - \sum_{x \in \{0,1\}^{n-1}} F_{S_i(p)}^{\mu^*,x}(q|v_i)g(x|a_i = 1) \tag{3}$$

as the probability that the residual supply faced by bidder i in equilibrium μ^* at price p is strictly higher than q conditional on having valuation v_i. Further, I let the function $w_i^* : [0,\overline{p}] \times [0,1] \times V \to \mathbb{R}_+$ satisfy

$$w_i^*(p,q|v_i) = \sum_{x \in \{0,1\}^{n-1}} f_{S_i(p)}^{\mu^*,x}(q|v_i)g(x|a_i = 1) \tag{4}$$

whenever the density $f_{S_i(p)}^{\mu^*,x}(q|v_i)$ of $F_{S_i(p)}^{\mu^*,x}(q|v_i)$ exists for all $x \in \{0,1\}^{n-1}$. Note that because, for any $x \in \{0,1\}^{n-1}$ and given $v_i \in V$, the density $f_{S_i(p)}^{\mu^*,x}(q|v_i)$ exists for a.e. $(p,q) \in [0,\overline{p}] \times [0,1]$, the function w_i^* is equal to the sum appearing on the right-hand side in (4) a.e. on $[0,\overline{p}] \times [0,1]$.

Because the action space is restricted to decreasing p_i^j and increasing q_i^j, bidders might submit less than the allowed number k of price-quantity pairs. A bidder for whom at least one of the restrictions binds submits a bid-schedule with either $p_i^j = p_i^{j+1}$, or $q_i^j = q_i^{j-1}$, or both, for at least one $j \in \{1,\dots,k\}$. I say that the bid-schedule of bidder i consists of ℓ_i distinguishable price-quantity pairs with $p_i^j > p_i^{j+1}$ and $q_i^j > q_i^{j-1}$, $j = 1,\dots,\ell_i$, where I have deleted the price-quantity pairs from the original bid-schedule for which equality holds either for the quantities, the prices, or both.

The next proposition follows from the optimality conditions with respect to the submitted quantities and with respect to the submitted prices of player i and from the observation made in the proof to Proposition 1 that ties happen with probability zero in equilibrium. Let $q_i^0 = 0$ and $(p_i^{\ell_i+1}, q_i^{\ell_i+1}) = (0,1)$, then

Proposition 2. *In equilibrium $\mu^* \in \mathscr{M}^n$, it holds for all bidders $i \in \{1,...,n\}$, all steps $j \in \{1,...,\ell_i\}$ and a.e. type $v_i \in V$ that the distribution of $S_i(p_i^j)$ is continuous in $q \in [q_i^{j-1}, q_i^j]$ and that the distinguishable price-quantity pairs $p_i^j, q_i^j > 0$ satisfy*

$$\left[v_i\left(q_i^j\right) - p_i^j\right] W_i^*(p_i^j, q_i^j | v_i) - \left[v_i\left(q_i^j\right) - p_i^{j+1}\right] W_i^*(p_i^{j+1}, q_i^j | v_i) \geq 0 \tag{5}$$

$$\int_{q_i^{j-1}}^{q_i^j} \left[\left[v_i(q) - p_i^j\right] w_i^*(p_i^j, q | v_i) - W_i^*(p_i^j, q | v_i)\right] \mathrm{d}q \geq 0, \tag{6}$$

where (5) *holds with equality iff $q_i^j < 1$ and* (6) *holds with equality iff $p_i^j < \overline{p}$.*

Equation (5) follows from the optimality conditions with respect to the quantity points submitted, and equation (6) follows from the optimality conditions with respect to the price points submitted. Equation (6) relies on the fact that the equilibrium distribution of $S_i(p_i^j)$ is continuous on $(q_i^{j-1}, q_i^j]$ for all $i \in \{1,...,n\}$ and $j \in \{1,...,\ell_i\}$, which is a direct consequence of the absence of ties in equilibrium.

While the optimality conditions with respect to quantities are also used in the characterization of Kastl (2012), the optimality conditions with respect to prices are not used in Kastl (2012). Hence, Proposition 2 extends the equilibrium characterization of Kastl (2012), and it is precisely this additional structure that I am going to exploit for the formulation of the bounds and the estimator of the number of best-response violations. Furthermore, the interpretation of the equilibrium share winning probability W_i^* in terms of the distribution of the residual supply function $S_i(p)$ will be crucial for the estimation.

In view of the empirical analysis to come, I conclude this section by considering the implications of two additional assumptions about the type space which will be fruitfully employed.

Definition 2 (Bidder independence). *Bidders $i \in M \subset \{1,...,n\}$ are called independent if both $a_i \in \{0,1\}$ and $v_i \in V$ are independently distributed across all bidders $i \in M$.*

Definition 3 (Bidder Symmetry). *Bidders $i \in M \subset \{1,...,n\}$ are called symmetric if all bidders $i \in M$ have the same marginal density η of valuations v_i and the same probability of being active $p = \mathbb{P}(a_i = 1)$.*

Suppose the set $\{1,...n\}$ of independent bidders can be partitioned into $m \leq n$ sets M_j, $j = 1,...,m$ of symmetric bidders with $n_j = |M_j|$ with participation probability p_j. I write $h_{ij}(x)$ for the probability that a bidder from set M_i is faced with $0 \leq x \leq n_j$ opponent bidders from set M_j. For $i \neq j$, we have

$$h_{ij}(x) = \binom{n_j}{x} p_j^x (1 - p_j)^{n_j - x},$$

and for $i = j$, we have

$$h_{ii}(x) = \binom{n_i - 1}{x} p_i^x (1 - p_i)^{n_i - 1 - x}.$$

Further, I denote by $F_{S_i(p)}^{\mu^*,(x_1,...x_m)}(q)$ the cumulative distribution function of the residual supply $S_i(p)$ faced by bidder i in equilibrium μ^* given that he is faced with x_j opponent bidders from set M_j, $j = 1,...,m$. Observe that because the bidders' valuations are independent, $F_{S_i(p)}^{\mu^*,(x_1,...x_m)}$ does not depend on bidder i's valuation v_i and consequently neither do W_i^* and w_i^*. Hence, I get

Lemma 2. *Assume that all bidders $i \in \{1,...,n\}$ are independent, that $\{M_1,...,M_m\}$ is a partition of the bidder set into $m \leq n$ sets of symmetric bidders, and that bidder i is in set M_j. Then, W_i^**

is independent of v_i and can be expressed as

$$W_i^*(q,p) = 1 - \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \cdots \sum_{x_j=0}^{n_j-1} \cdots \sum_{x_m=0}^{n_m} h_{i1}(x_1) \times \ldots \times h_{im}(n_m) F_{S_i(p)}^{\mu^*,(x_1,\ldots x_m)}(q), \tag{7}$$

whereas w_i^*, too, is independent of v_i and satisfies

$$w_i^*(q,p) = \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \cdots \sum_{x_j=0}^{n_j-1} \cdots \sum_{x_m=0}^{n_m} h_{i1}(x_1) \times \ldots \times h_{im}(n_m) f_{S_i(p)}^{\mu^*,(n_1,\ldots n_m)}(q) \tag{8}$$

whenever the density $f_{S_i(p)}^{\mu^*,(x_1,\ldots x_m)}(q)$ of $F_{S_i(p)}^{\mu^*,(x_1,\ldots x_m)}(q)$ exists for all $x_j \leq n_j$ and all $j = 1,\ldots,m$.

3 Bounds

This section presents an upper and a lower bound on the valuations that satisfy the characterization (6) in Proposition 2. To do so, I employ the following additional assumption on the type space that holds throughout the rest of the paper.

(A4) All bidders $i \in \{1,\ldots,n\}$ are independent.

As seen above, it follows from Assumption (A4) that both W_i^* and w_i^* are independent of v_i for any bidder i. For a bidder i and a step $j \in \{1,\ldots,\ell_i\}$, let V_i^j be the set of non-increasing functions $v_i^j : [q_i^{j-1}, q_i^j] \to [0,\bar{v}]$, and denote by

$$V_0^j \equiv \left\{ v_i \in V_i^j : \int_{q_i^{j-1}}^{q_i^j} \left[\left[v_i(q) - p_i^j \right] w_i^*(p_i^j, q) - W_i^*(p_i^j, q) \right] \mathrm{d}q = 0 \right\} \tag{9}$$

the set of marginal valuation functions satisfying the optimality conditions (6) on the respective interval $[q_i^{j-1}, q_i^j]$ for $p_i^j \in (0,\bar{p})$. For the following, I employ the pointwise partial order on the set V_i^j.

Definition 4 (Order on V_i^j)**.** *Let $v_i, \tilde{v}_i \in V_i^j$. We have $v_i \geq \tilde{v}_i$ iff $v_i(q) \geq \tilde{v}_i(q)$, $\forall q \in [q_i^{j-1}, q_i^j]$.*

The aim of the following is to characterize functions that lie weakly above the least upper bound $\overline{v}_0^j = \vee(V_0^j)$ and weakly below the greatest lower bound $\underline{v}_0^j = \wedge(V_0^j)$ of the set V_0^j. I construct such functions as a fixed point of an auxiliary function on $V_i^j \times V_i^j$ that in turn requires knowledge of two functions lying above $\overline{v}_0^j$ and below $\underline{v}_0^j$. While I assume knowledge of such functions in the derivation below, I will present an empirical strategy of finding such functions later on.

Let $\overline{\omega}^j : [q_i^{j-1}, q_i^j] \to \mathbb{R}_+$ be a non-increasing function satisfying $\overline{\omega}^j(q) \geq \overline{v}_0(q)$ for all $q \in [q_i^{j-1}, q_i^j]$, and let $\underline{\omega}^j : [q_i^{j-1}, q_i^j] \to \mathbb{R}_+$ be a non-increasing function satisfying $\underline{\omega}^j(q) \leq \underline{v}_0(q)$ for all $q \in [q_i^{j-1}, q_i^j]$. For $q \in [q_i^{j-1}, q_i^j]$, $v \in [0,\overline{v}]$ and $v_u^j, v_l^j \in V_i^j$, $j \in \{1,\ldots,\ell_i\}$, I define

$$\mu_u(q,v,v_l^j) = \int_{q_i^{j-1}}^{q} \left[\left[\max\left\{ v, v_l^j(x) \right\} - p_i^j \right] w_i^*(p_i^j, x) - W_i^*(p_i^j, x) \right] \mathrm{d}x$$
$$+ \int_{q}^{q_i^j} \left[\left[v_l^j(x) - p_i^j \right] w_i^*(x, p_i^j) - W_i^*(p_i^j, x) \right] \mathrm{d}x \tag{10}$$

and

$$\mu_l(q,v,v_u^j)=\int_{q_i^{j-1}}^{q}\Big[\Big[v_u^j(x)-p_i^j\Big]w_i^*(p_i^j,x)-W_i^*(p_i^j,x)\Big]\mathrm{d}x$$
$$+\int_{q}^{q_i^j}\Big[\Big[\min\big\{v,v_u^j(x)\big\}-p_i^j\Big]w_i^*(p_i^j,x)-W_i^*(p_i^j,x)\Big]\mathrm{d}x. \quad (11)$$

Clearly, both μ_u and μ_l are continuously decreasing in v. Using (10)–(11), I let the pair of functions (g_u,g_v) with $g_u:V_i^j\to V_i^j$ and $g_l:V_i^j\to V_i^j$ be given for $q\in[q_i^{j-1},q_i^j]$ as

$$g_u(v_l^j)(q)=\begin{cases}\underline{\omega}^j(q) & \text{if } \mu_u(q,\underline{\omega}^j(q),v_l^j)>0\\ \overline{\omega}^j(q) & \text{if } \mu_u(q,\overline{\omega}^j(q),v_l^j)<0\\ \min\{v\in[\underline{\omega}^j(\tilde{q}),\overline{\omega}^j(\tilde{q})]:\mu_u(q,v,v_l^j)\geq 0\} & \text{else}\end{cases} \quad (12)$$

and

$$g_l(v_u^j)(q)=\begin{cases}\underline{\omega}^j(q) & \text{if } \mu_l(q,\underline{\omega}^j(q),v_u^j)>0\\ \overline{\omega}^j(q) & \text{if } \mu_l(q,\overline{\omega}^j(q),v_u^j)<0\,.\\ \max\{v\in[\underline{\omega}^j(\tilde{q}),\overline{\omega}^j(\tilde{q})]:\mu_l(q,v,v_u^j)\leq 0\} & \text{else}\end{cases} \quad (13)$$

The function g_u returns for any given $v_j^l\in V_i^j$ a function in V_i^j that returns at any $q\in[q_i^{j-1},q_i^j]$ the highest point such that there is a lowest non-increasing function going through that point that still satisfies the characterization (6) and is higher than v_j^l. Conversely, the function g_l returns for any given $v_j^u\in V_i^j$ a function in V_i^j that returns at any $q\in[q_i^{j-1},q_i^j]$ the lowest point such that there is a highest non-increasing function going through that point that still satisfies the characterization (6) and is lower than v_j^u.

One would suspect that mutually consistent such functions form upper and lower bounds on the set V_0^j. The following paragraphs show that this intuition is indeed correct. Let $v_{lu}^j=(v_l^j,v_u^j)\in V_i^j\times V_i^j$ and $g=(g_u,g_l)$, and consider the set of fixed points

$$\Phi(g)\equiv\Big\{v\in V_i^j\times V_i^j:v=g(v)\Big\}. \quad (14)$$

I start by establishing that the set $\Phi(g)$ is a complete lattice when we put the following partial order on the set $V_i^j\times V_i^j$.

Definition 5 (Order on $V_i^j\times V_i^j$). *Let $v_{lu}^j,\tilde{v}_{lu}^j\in V_i^j\times V_i^j$. We have $v_{lu}^j\geq\tilde{v}_{lu}^j$ iff both $v_u^j\leq\tilde{v}_u^j$ and $v_l^j\geq\tilde{v}_l^j$ hold.*

Lemma 3. *The set $\Phi(g)$ is a complete lattice.*

The proof to Lemma 3 relies on the Knaster-Tarski fixed point theorem. By Lemma 3, there exist, for any bidder i and step $j\in\{1,\ldots,\ell_i\}$, a least v_{lu}^j solving $v_{lu}^j=g(v_{lu}^j)$. The next proposition states that this least fixed point is below the least upper and greatest lower bounds of V_0^j, as depicted in Figure 1.

Proposition 3. *It holds that $\wedge(\Phi(q))\leq(\underline{v}_0^j,\overline{v}_0^j)$.*

Knowing that the set $\Phi(q)$ has a least fixed-point suggests a numerical strategy to find that least fixed point by a simple iteration procedure of g that has $(\underline{\omega}^j,\overline{\omega}^j)$ as initial condition. In Section 5, I present such a fixed point iteration algorithm and in Section 6 I discuss strategies to derive feasible initial conditions $(\underline{\omega}^j,\overline{\omega}^j)$.

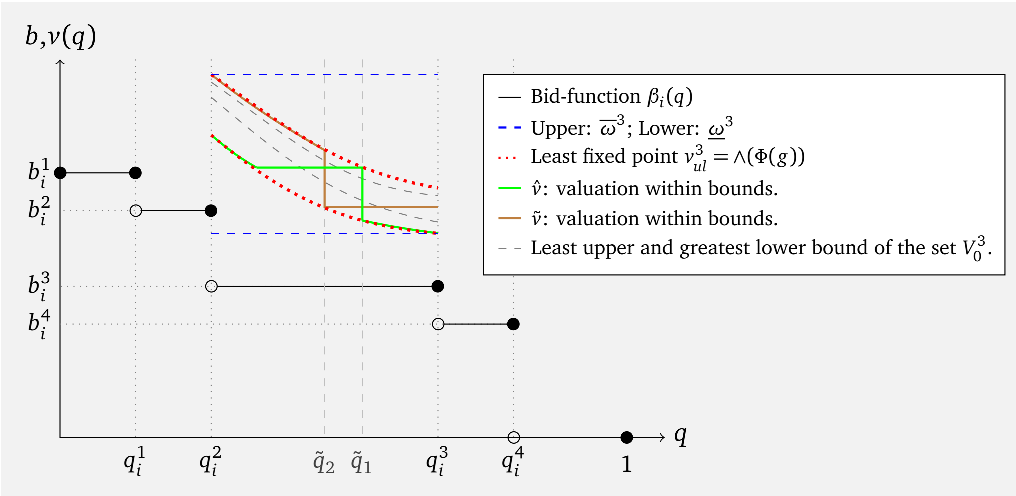

Figure 1: *The figure depicts in black a bid-function $\beta_i(q)$ for bidder i with four steps, and in dotted red the least fixed point $v_{lu}^3 = \wedge(\Phi(g))$ for step $j = 3$, where Φ is given in* (14)*. The functions v_{lu}^3 are lower and upper bounds of V_0^3 whose upper and lower bounds are depicted in dashed gray. The valuation function $\hat{v}$ ($\tilde{v}$) is the lowest (highest) non-increasing valuation functions that is equal to $v_l^3(q)$ for $q > \tilde{q}_1$ ($q \leq \tilde{q}_2$) and for which the right-hand side of* (6) *is greater (lower) zero. By the construction of g_u, g_l as given in* (12) *and* (13)*, $\hat{v}$ (resp. $\tilde{v}$) satisfies $\hat{v}(\tilde{q}_1) = v_u^3(\tilde{q}_1)$ (resp. $\tilde{v}(\tilde{q}_2) = v_l^3(\tilde{q}_2)$).*

4 Estimation and Consistency

I now turn to the estimation of W_i^* and w_i^*. I assume that the data at hand covers a series of auctions $t = 1, ..., T$ with $n_t \geq 2$ participating bidders in auction t. The submitted bid-schedules are collected in the set $\mathscr{B}$ with cardinality $|\mathscr{B}| = \sum_{t=1}^{T} n_t$. Potential bidders have to register prior to bidding, and the number of potential bidders is $N \geq \min\{n_1, ..., n_t\}$. Throughout the following I make the following assumptions on the data generating process in addition to (A1)-(A2) and (A4).

(A5) All bidders $i \in \{1, ..., N\}$ are symmetric. In particular it holds for all bidders that the probability to participate is $p = \mathbb{P}(a_i = 1)$.

(A6) All auctions $t \in \{1, ..., T\}$ are identical, that is, the valuation and participation distributions are identical across all auctions.

Assumptions (A5) and (A6) can be weakened: Assumption (A5) can be weakened to assuming that there are subsets of symmetric agents, and Assumption (A6) can be weakened to assuming that there are subsets of comparable auctions. For the sake of expositional clarity I focus on the simplest case in which both Assumptions (A5) and (A6) hold. Nevertheless I discuss possibilities to weaken (A5)–(A6) as we go along, and in the application in Section 6 I assess several of these possibilities.

4.1 Two simple estimators

As all bidders are symmetric and independent and all auctions are identical, every bidder i is faced with the same distribution of the residual supply function function in every auction that he is active. Hence, dropping the subscript i in the following, we have by Lemma 2, for a.e. $(p,q) \in (0,\overline{p}) \times (0,1)$,

$$w^*(p,q) = \sum_{n=0}^{N-1} f_{S(p)}^{\mu^*,n}(q) h(n). \tag{15}$$

The right-hand side of (15) consists of the sum of the densities $f_{S(p)}^{n}$ of the value of the residual supply function $S(p)$ at q given that the bidder at hand faces n opponents, that are weighted by the probability $h(n)$ of having exactly n opponents. The probability $h(n)$ that a bidder is faced with $0 \le n \le N-1$ opponent bidders in an auction can be estimated by

$$\hat{h}(n) = \binom{N-1}{n} \hat{p}^n (1-\hat{p})^{N-1-n}, \tag{16}$$

where $\hat{p}$ is an estimate of the parameter p obtained by

$$\hat{p} = \frac{1}{T} \sum_{t=1}^{T} \frac{n_t}{N}. \tag{17}$$

On the other hand, the density $f_{S(p)}^{\mu^*,n}(q)$ is estimated with

$$\hat{f}_{S(p)}^{\mu^*,n}(q) = \frac{1}{|\mathscr{B}|^n} \sum_{\beta_1 \in \mathscr{B}} \cdots \sum_{\beta_n \in \mathscr{B}} K_b \left(\sum_{j=1}^{n} \beta_j^{-1}(p) - q \right)$$

where $K_b(.)$ stands for an arbitrary kernel with bandwidth $b > 0$ and we have $\hat{f}_{S(p)}^{\mu^*,0}(q) = 0$ for all $q \in (0,1)$ and $p \in (0,\bar{p})$ by definition. Taken together, I define

$$\hat{w}^*(p,q) = \sum_{n=0}^{N-1} \hat{f}_{S(p)}^{\mu^*,n}(q) \hat{h}(n) \tag{18}$$

for all $(p,q) \in (0,\overline{p}) \times (0,1)$. As regards the cumulative distribution of the received quantity, I also skip the subscript referring to bidder i and write

$$W^*(p,q) = 1 - \sum_{n=0}^{N-1} F_{S_i(p)}^{\mu^*,n}(q) h(n).$$

In order to estimate $W^*(q,b)$, I use $\hat{h}(n)$ as in (16) above to estimate $h(n)$, and estimate $F_{S_i(p)}^{\mu^*,n}(q)$ with

$$\hat{F}_{S(p)}^{\mu^*,n}(q) = \frac{1}{|\mathscr{B}|^n} \sum_{\beta_1 \in \mathscr{B}} \cdots \sum_{\beta_n \in \mathscr{B}} \mathbb{1} \left\{ \sum_{j=1}^{n} \beta_j^{-1}(p) \ge 1 - q \right\},$$

with $\hat{F}_{S(p)}^{\mu^*,0}(q) = 0$ for all $q \in (0,1)$ and $p \in (0,\bar{p})$ by definition. Taken together, we have

$$\hat{W}^*(p,q) = 1 - \sum_{n=0}^{N-1} \hat{F}_{S(p)}^{\mu^*,n}(q) \hat{h}(n). \tag{19}$$

As regards consistency of the two estimators (18) and (19) above, first observe that we have $\text{plim}_{T \to \infty} \hat{h}(n) = h(n)$ because $\text{plim}_{T \to \infty} \hat{p} = p$ by the law of large numbers (cf. van der Vaart, 1998, Chapter 19). Secondly, note that from $T \to \infty$, it follows that $|\mathscr{B}| \to \infty$. Consequently, we have $\text{plim}_{T\to\infty} \hat{F}^{\mu^*,n}_{S(p)}(q) = F^{\mu^*,n}_{S(p)}(q)$ for all $q \in (0,1)$ and $p \in (0,\overline{p})$ and $n \in \{1,...,N\}$ again by the law of large numbers (cf. van der Vaart, 1998, Chapter 19), which together with the consistency of $\hat{h}(n)$ gives us consistency of $\hat{W}^*$.

Furthermore, for any sequence of kernels with bandwidth $b \to 0$ satisfying $b|\mathscr{B}|^{n-1} \to \infty$ as $|\mathscr{B}| \to \infty$ for all $n \in \{1,...,N\}$ it follows for any $n \in \{1,...,N\}$ and for a.e. $q \in (0,1)$ and $p \in (0,\overline{p})$ that $\text{plim}_{T\to\infty} \hat{f}^{n}_{S(p)}(q) = f^{n}_{S(p)}(q)$ whenever $f^{n}_{S(p)}$ is continuous (cf. Parzen, 1962, Corollary 1A and Theorem 2A), which together with the consistency of $\hat{h}(n)$ gives us consistency of $\hat{w}^*$.[2] The next lemma summarizes these findings.

Lemma 4 (Consistency). *For all $q \in (0,1)$ and $p \in (0,\overline{p})$, we have*

$$\underset{T\to\infty}{plim}\ \hat{W}^*(p,q) = W^*(p,q),$$

and for a.e. $q \in (0,1)$ and $p \in (0,\overline{p})$ and any sequence of kernels with bandwith $\lim_{T\to\infty} b(T) = 0$ *satisfying* $\lim_{T\to\infty} b(T)|\mathscr{B}|^{n-1} = \infty$ *for all $n \in \{1,...,N\}$ we have*

$$\underset{T\to\infty}{plim}\ \hat{w}^*(p,q) = w^*(p,q).$$

I refrain from deriving an expression of the asymptotic variance of $\hat{w}^*$ and $\hat{W}^*$, but note that it can be approximated by drawing bootstrap samples of the original data and running with these bootstrap samples the resampling procedure as discussed in Section 5 (cf. also Hortaçsu and McAdams, 2010) .

4.2 Best response violations

Using the assumption of decreasing marginal valuations and combining the characterizations (5)–(6) from Proposition 2, I obtain the following result.

Lemma 5. *In the equilibrium μ^* of auction t, the submitted price-quantity pairs (p_i^j, q_i^j) satisfy, for every bidder $i = 1,...,n_t$, a.e. $v_i \in V$, and steps $j \in \{1,...,\ell_i - 1\}$*

$$\frac{\int_{q_i^{j-1}}^{q_i^j} W_i^*(p_i^j,q)\mathrm{d}q}{\int_{q_i^{j-1}}^{q_i^j} w_i^*(p_i^j,q)\mathrm{d}q} \frac{W_i^*(p_i^j,q_i^j) - W_i^*(p_i^{j+1},q_i^j)}{W_i^*(p_i^{j+1},q_i^j)} \geq$$

$$p_i^j - p_i^{j+1} \geq \frac{\int_{q_i^{j}}^{q_i^{j+1}} W_i^*(p_i^{j+1},q)\mathrm{d}q}{\int_{q_i^{j}}^{q_i^{j+1}} w_i^*(p_i^{j+1},q)\mathrm{d}q} \frac{W_i^*(p_i^j,q_i^j) - W_i^*(p_i^{j+1},q_i^j)}{W_i^*(p_i^j,q_i^j)}. \quad (20)$$

[2]To be more precise, the unscaled kernel $K(x) = bK_b(bx)$ must satisfy $\sup_x |K(x)| < \infty$, $\int |K(x)|\mathrm{d}x < \infty$, $\int K(x)\mathrm{d}x = 1$, and $\lim_{x\to\infty} |xK(x)| = 0$ (cf. Assumptions (1.11)-(1.13) together with Corollary 1A in Parzen, 1962).

The derivation of above inequalities becomes evident in the following paragraphs. Plugging the estimators $\hat{w}_i^*$ and $\hat{W}_i^*$ into the inequalities reported in Lemma 5 above, I define, for all $p_i^j \in (0, \bar{p})$ and $q_i^j \in (0,1)$,

$$\theta_i^{j+} = \max\left\{ \frac{\hat{W}_i^*(p_i^j, q_i^j) - \hat{W}_i^*(p_i^{j+1}, q_i^j)}{\hat{W}_i^*(p_i^j, q_i^j)} \frac{\int_{q_i^j}^{q_i^{j+1}} \hat{W}_i^*(p_i^{j+1}, q)\mathrm{d}q}{\int_{q_i^j}^{q_i^{j+1}} \hat{w}_i^*(p_i^{j+1}, q)\mathrm{d}q} - \left(p_i^j - p_i^{j+1}\right), 0 \right\}$$

and

$$\theta_i^{j-} = \max\left\{ \left(p_i^j - p_i^{j+1}\right) - \frac{\hat{W}_i^*(p_i^j, q_i^j) - \hat{W}_i^*(p_i^{j+1}, q_i^j)}{\hat{W}_i^*(p_i^{j+1}, q_i^j)} \frac{\int_{q_i^{j-1}}^{q_i^j} \hat{W}_i^*(p_i^j, q)\mathrm{d}q}{\int_{q_i^{j-1}}^{q_i^j} \hat{w}_i^*(p_i^j, q)\mathrm{d}q}, 0 \right\}.$$

Both θ_i^{j+} and θ_i^{j-} detect violations of best response behavior by checking whether the point estimate of $v_i(q_i^j)$ given by

$$\hat{v}_i(q_i^j) = p_i^j + \left(p_i^j - p_i^{j+1}\right) \frac{\hat{W}_i^*(p_i^{j+1}, q_i^j)}{\hat{W}_i^*(p_i^j, q_i^j) - \hat{W}_i^*(p_i^{j+1}, q_i^j)}, \tag{21}$$

obtained by using the equilibrium bid-schedule characterization (5), lies between the estimated upper bound $\hat{v}_i^u(q_i^j)$ given by

$$\hat{v}_i^u(q_i^j) = p_i^j + \left[\int_{q_i^{j-1}}^{q_i^j} \hat{W}_i^*(p_i^j, q)\mathrm{d}q \right] \left[\int_{q_i^{j-1}}^{q_i^j} \hat{w}_i^*(p_i^j, q)\mathrm{d}q \right]^{-1}, \tag{22}$$

and the estimated lower bound $\hat{v}_i^l(q_i^j)$ given by

$$\hat{v}_i^l(q_i^j) = p_i^{j+1} + \left[\int_{q_i^j}^{q_i^{j+1}} \hat{W}_i^*(p_i^{j+1}, q)\mathrm{d}q \right] \left[\int_{q_i^j}^{q_i^{j+1}} \hat{w}_i^*(p_i^{j+1}, q)\mathrm{d}q \right]^{-1}, \tag{23}$$

where both $\hat{v}_i^u(q_i^j)$ and $\hat{v}_i^l(q_i^j)$ are obtained by using the equilibrium bid-schedule characterization (6) and the assumption of decreasing marginal valuations. If $\theta_i^{j+} = \theta_i^{j-} = 0$, then the the hypothesis of best response behavior cannot be rejected, but if either θ_i^{j+} or θ_i^{j-} is non-zero then we have a violation of best response behavior for price-quantity pair (p_i^j, q_i^j). Observe that the last step $j = \ell_i$ never violates best response behavior for any bidder i. Aggregation of θ_i^{j+} and θ_i^{j-} for a given auction t is possible in several ways, one of which is given by

$$\Theta_t = \left[\sum_{i=1}^{n_t} \sum_{j=1}^{\ell_i - 1} \left[\mathbb{1}\{(\theta_i^{j+} > 0) \vee (\theta_i^{j-} > 0)\} \right] \right] \left[\sum_{i=1}^{n_t} \ell_i \right]^{-1}, \tag{24}$$

where $\vee$ denotes logical disjunction. $\Theta_t \in [0,1]$ estimates the fraction of violations of the inequalities stated in Lemma 5 in the submitted price-quantity pairs for a given auction t. The interpretation of Θ_t is intuitive: Because the inequalities given in Lemma 5 are necessary for best response behavior, Θ_t returns a lower bound on the fraction of price-quantity pairs that violate best response behavior given that we can consistently estimate the equilibrium distribution of $S_i(p)$ for any bidder i.

This allows for a comparison either between the degree of irrationality in different auctions or between the quality of different estimators of W_i^* and w_i^*: If we are confident enough that the estimates of W_i^* and w_i^* are accurate, then the values of Θ_t can be interpreted as the degree at which the bidders violate their best responses. So, if we have a series of auctions t, then changes in the values of Θ_t in these auctions might be taken as an indicator for whether the bidders learn to play best response over time or not. If, on the other hand, we are not confident that the estimates of W_i^* and w_i^* are accurate, then we may take the value of Θ_t as an indicator for how many best-response violations we have to allow for when using $\hat{w}_i^*$ and $\hat{W}_i^*$. That is, we can take Θ_t as an indicator for the accuracy of the assumptions that the estimators of W_i^* and w_i^* make on the type space. Comparing different type space assumptions by means of the corresponding values of Θ_t thus yields an ordering for the validity of these assumptions based on the degree of irrationality that has to be assumed so that the data is consistent with the model.

In the application in Section 6 below, I follow the second possibility. Again, I refrain from deriving an analytical expression for the variance of Θ_t but note that it can be computed by standard bootstrap methods as discussed e.g. in Hortaçsu and McAdams (2010).

5 Implementation

5.1 Estimating W_i^* and w_i^*

A direct calculation of $\hat{w}^*(q,p)$ or $\hat{W}^*(q,p)$ is unfeasible already for a small number $|\mathscr{B}|$ of bid functions. For this reason, I employ a resampling procedure to approximate both $\hat{w}^*(q,p)$ and $\hat{W}^*(q,p)$. The procedure repeatedly draws at random from the set $\mathscr{B}$ of available bid-functions a sample whose size is binomially distributed with parameters $N-1$ and $\hat{p}$ and then constructs the resulting residual supply function. This yields a set of residual supply functions from which the distribution of the residual supply $S(p)$ at the prices $p \in (0,\bar{p})$ of interest can be estimated. The details are given in Algorithm 1: The number of resampled supply functions is denoted by $R > 0$ (where R is chosen sufficiently large), the set of bid schedules is $\mathscr{B}$, the list of the number n_t of participants in the auctions $t = 1,...,T$ is $\{n_t\}_{t=1,...,T}$, and the total number of registered bidders is N.

Algorithm 1 Resampling $\hat{W}^*$ and $\hat{w}^*$

Require: $R, \mathscr{B}, N, \{n_t\}_{t=1,...,T}$.

Estimate p with $\hat{p} = T^{-1}\sum_{t=1}^{T} n_t/N$.
$r \leftarrow 1$
for $r = 1$ to R **do**
    Draw the number of opponent bidders $n \sim B(N-1,\hat{p})$
    Randomly draw a set of n bid functions β from $\mathscr{B}$ with replacement.
    Compute the residual supply function $S_r(p) = 1 - \sum_{i=1}^{n-1} \beta_i^{-1}(p)$.
end for
return $\hat{w}^{*,R}(q,p) = R^{-1}\sum_{r=1}^{R} K_h(S_r(p)-q)$ and $\hat{W}^{*,R}(q,p) = R^{-1}\sum_{r=1}^{R} \mathbb{1}\{S_r(p) \geq q\}$

Algorithm 1 is inspired by the resampling procedures used in Hortaçsu and McAdams (2010) and Kastl (2011), but differs from these in two important respects: First, both Hortaçsu and McAdams (2010) and Kastl (2011) assume fixed participation while Algorithm 1 incorporates random participation. Second, both algorithms in Hortaçsu and McAdams (2010) and Kastl (2011) compute the empirical distribution function of the quantity received q^c for a given sample of bid-schedules conditional on a bid-schedule b_i whereas Algorithm 1 returns estimates of the distribution of the residual supply function $S(p)$ faced by the bidders.

As R grows large, the resampling estimator $\hat{w}^{*,R}$ approaches $\hat{w}^*$ and the resampling estimator $\hat{W}^{*,R}$ approaches $\hat{W}^*$ (see Kastl, 2011, for a discussion). As long as R and T are finite however, the resampling estimators $\hat{w}^{*,R}$ and $\hat{W}^{*,R}$ will have non-zero variance. The variance in the resampling estimators comes from two sources: First, there is an error with respect to the approximated estimators $\hat{w}^*$ and $\hat{W}^*$, and, second, there is an error stemming from the fact that $\hat{w}^*$ and $\hat{W}^*$ themselves have non-zero variance. Because we are not directly interested in $\hat{w}^*$ and $\hat{W}^*$, however, I will not report these variances. Rather, I will follow the general approach in the literature to report the bootstrap confidence intervals for the estimators of interest that use $\hat{w}^*$ and $\hat{W}^*$ – which are the estimator of Θ_t and that of the least fixed point of g. The confidence bands are calculated by rerunning the resampling algorithm on bootstrap bid-function samples of the original sample (cf. Hortaçsu and McAdams, 2010; Kastl, 2011) and reporting the respective quantiles of the estimates thus obtained.

Algorithm 1 can be modified such as to accommodate for the assumption that there are subsets of auctions with symmetrical type spaces, in which case we would restrict the set $\mathscr{B}$ of bid-schedules to the respective subset when approximating $\hat{W}_i^*$ and $\hat{w}_i^*$ for a bidder in that subset. Furthermore, drawing from the different subsets according to a distribution that is estimated based on observable covariates of the auctions in the subsets, along the lines in Hortaçsu and McAdams (2010), would be straightforward to implement. Last, the algorithm could also accommodate the assumption that the set of bidders is partitioned into subsets of symmetrical bidders, in which case it would first estimate the participation probability for the bidders in the respective subsets, and then construct approximations of $\hat{W}_i^*$ and $\hat{w}_i^*$ separately for the bidders in the respective subsets.

5.2 Computing the Bounds

It remains to present the algorithm to compute the fixed point of g. For a given step $j \in \{1,...,\ell_i\}$, I partition the line segment $[q_i^{j-1}, q_i^j]$ into $h_q > 0$ steps and denote the partition by

$$\mathcal{Q} = \left\{ q_i^{j-1}, q_i^{j-1} + \frac{q_i^j - q_i^{j-1}}{h_q}, q_i^{j-1} + 2\frac{q_i^j - q_i^{j-1}}{h_q}, ..., q_i^j \right\}, \tag{25}$$

with the j-th element denoted by $(\mathcal{Q})_j$. I define step-functions $\tilde{v}_l(.,j,x,w)$ and $\tilde{v}_u(.,j,x,w)$ for $x \in [q_i^{j-1}, q_i^j]$, $j \in \{1,...,h_q+1\}$, and $w \in V_i^j$ on $[q_i^{j-1}, q_i^j]$ given by

$$v_u(q,j,x,w) = \sum_{i=1}^{h_q} v_{u,i} \cdot \mathbb{1}\{q \in [(\mathcal{Q})_i, (\mathcal{Q})_{i+1})\},$$

$$\text{with } \tilde{v}_{u,i} = \begin{cases} \max\{x, w((\mathcal{Q})_i) & \text{for } i \leq j \\ w((\mathcal{Q})_i) & \text{for } i > j \end{cases}, \tag{26}$$

and by

$$v_l(q,j,x,w) = \sum_{i=1}^{h_q} v_{l,i} \cdot \mathbb{1}\{q \in [(\mathcal{Q})_i, (\mathcal{Q})_{i+1})\},$$

$$\text{with } \tilde{v}_{l,i} = \begin{cases} w((\mathcal{Q})_i) & \text{for } i \leq j \\ \min\{x, w((\mathcal{Q})_i)\} & \text{for } i > j \end{cases}. \tag{27}$$

Plugin in (26) and (27) into

$$\Phi(v) = \int_{q_i}^{q_i^{j+1}} \left[\left[v(q) - p_i^j\right]\hat{w}^{*,R}(p_i^j, q) - \hat{W}^{*,R}(p_i^j, q)\right] j \mathrm{d}q, \tag{28}$$

then gives me the discrete analogue to the functions μ_u and μ_l given in (10) and (11), respectively.

The procedure to iterate the fixed point of g for step $j \in \{1, ..., \ell_i\}$ of bidder i with bid-schedule b_i is described with Algorithm 2. The algorithm requires the number of steps h_q, the partition $\mathcal{Q}$, and a pair of functions $(\overline{\omega}^j, \underline{\omega}^j)$ that satisfy $\overline{\omega}^j > \overline{v}_0^j$ and $\underline{\omega}^j < \underline{v}_0^j$. The algorithm then iterates (g_l, g_u) starting with arguments $(u_1, l_1) = (\overline{\omega}^j, \underline{\omega}^j)$ and constructing sequences $l_r \in V_i^j$ and $u_r \in V_i^j$, $r = 2, ...$ until the sum of differences in the respective function values between two iterations r and $r+1$ under the sup-norm is lower than some pre-specified $\epsilon > 0$.[3] The algorithm hinges crucially on the fact that the initial conditions $(\overline{\omega}^j, \underline{\omega}^j)$ indeed satisfy $\overline{\omega}^j > \overline{v}_0^j$ and $\underline{\omega}^j < \underline{v}_0^j$. I discuss the derivation of such initial conditions in the application to which we turn next.

6 Application: Swiss Meat Import Rights Auctions

The aim of this section is to give proofs of concept both for the estimator of the share of best response violations and for the algorithm to construct bounds on the bidders' valuations between the submitted quantities. I do so by looking at data from high-quality beef (HQB) import quota auctions to Switzerland in which bidders compete for the right to import a fixed, divisible quantity of high-quality beef at a low tariff.

6.1 Data: Swiss High-Quality Beef Import Quota Auctions

Meat imports to Switzerland are subject to a two-part tariff: for each of the different meat categories that the meat market is segmented in there is a monthly or quarterly quota that the market participants are allowed to import at a low in-quota tariff. Once the quota is filled unlimited additional meat can be imported at a considerably higher out-of-quota tariff. From 2008 – 2015, the entire quotas in any of the categories were allocated through discriminatory share auctions; in 2015, the share of the respective quotas that are allocated by auction was reduced to 50%.[4]

[3] An alternative to the iteration procedure in Algorithm 2 is, of course, to directly iterate the composite function $g^l \circ g^u$ with initial condition $\underline{\omega}^j$. This will yield the lowest v_l^j for which it holds that $v_l^j = g^l(g^u(v_l^j))$, which then in turn allows to compute the highest v_u^j for which it holds that $v_u^j = g^u(g^l(v_u^j))$ by evaluating $g^u(v_l^j)$. I have chosen the formulation in Algorithm 2 because it I believe that it is notationally more transparent.

[4] This was due to pressure from agricultural and meat processing lobbying groups. The remaining quota is since allocated on the basis of domestic purchase, which is the allocation mechanism used before 2008.

Algorithm 2 Computing the bounds

Require: ϵ, h_q, $\mathscr{Q}$, $\overline{\omega}^j$, $\underline{\omega}^j$

$r \leftarrow 0$

$u_1 \leftarrow \overline{\omega}^j$

$l_1 \leftarrow \underline{\omega}^j$

repeat

  $r \leftarrow r+1$

  for $j = 1$ to h_q **do**

$$l_{r+1,j} \leftarrow \begin{cases} \underline{\omega}^j((\mathscr{Q})_j) & \text{if } \Phi(v_l(q,j,\underline{\omega}^j((\mathscr{Q})_j),u_r)) > 0 \\ \overline{\omega}^j((\mathscr{Q})_j) & \text{if } \Phi(v_l(q,j,\overline{\omega}^j((\mathscr{Q})_j),u_r)) < 0 \\ \min\{x \in [\underline{\omega}^j((\mathscr{Q})_j), \overline{\omega}^j((\mathscr{Q})_j)] : \\ \qquad \Phi(v_l(.,j,x,u_r)) \leq 0\} & \text{else} \end{cases}$$

$$u_{r+1,j} \leftarrow \begin{cases} \underline{\omega}^j((\mathscr{Q})_j) & \text{if } \Phi(v_l(.,j,\underline{\omega}^j((\mathscr{Q})_j),u_r)) > 0 \\ \overline{\omega}^j((\mathscr{Q})_j) & \text{if } \Phi(v_l(.,j,\overline{\omega}^j((\mathscr{Q})_j),u_r)) < 0 \\ \max\{x \in [\underline{\omega}^j((\mathscr{Q})_j), \overline{\omega}^j((\mathscr{Q})_j)] : \\ \qquad \Phi(v_u(.,j,x,u_r)) \geq 0\} & \text{else} \end{cases}$$

  end for

  $l_{r+1}(q) \leftarrow \sum_{i=1}^{h_q} l_{r+1,i} \cdot \mathbb{1}\{q \in [(\mathscr{Q})_i, (\mathscr{Q})_{i+1})\}$, $q \in [q_i^{j-1}, q_i^j]$

  $u_{r+1}(q) \leftarrow \sum_{i=1}^{h_q} u_{r+1,i} \cdot \mathbb{1}\{q \in [(\mathscr{Q})_i, (\mathscr{Q})_{i+1})\}$, $q \in [q_i^{j-1}, q_i^j]$

until $\sup_{q \in [q_i^j, q_i^{j+1}]} |l_r(q) - l_{r+1}(q)| + \sup_{q \in [q_i^j, q_i^{j+1}]} |u_r(q) - u_{r+1}(q)| \leq \epsilon$

return l_{r+1}, u_{r+1}

For each meat category, online-auctions are conducted by the Ministry of Agriculture. In general the quotas vary over time. The quotas are set and announced prior to the auction by the Ministry of Agriculture that usually follows the recommendation of the trade association of Swiss meat producers (*proviande*).[5] Bidding is open to all residents of Switzerland, but a registration at the Ministry of Agriculture is required. After every auction, the Ministry of Agriculture publishes quantities received and prices paid by every bidder. Import rights are valid until the next auction takes place.

I focus on the import quota auctions of high-quality beef (HQB). Auctions in this category take place roughly monthly, and bidders are allowed submit at most 5 price-quantity bids. The data set at hand covers 39 auctions from 2008 to 2010 with a total of 123 registered bidders. For every auction, the data contain the quota and the price-quantity bids of the active bidders. The quotas range from 67.5 tons to 630 tons per auction with a mean quota of 311.5 tons. Auction revenues per kg range from CHF 4.88 to CHF 14.79 with a mean of CHF 8.91 per kilogram. Absolute revenues per auction range from CHF 661,000 to CHF 7,984,000 with a mean revenue per auction of CHF 2,763,000. The accumulated revenue over the 39 auctions from 2008 to 2010 at hand amounts to CHF 107 million.

The number of active bidders varies between 58 and 82. The average number of active bidders in an auction is 70. The average share of successful bidders (that is of bidders having received a non-zero share) is 64% per auction. The lowest ratio between successful and active participants lies at 3 to 68, the highest ratio at 76 to 78. Stop-out prices range from CHF 3.21 per kg to CHF 14.41 per kilogram. The shares of the total quota allocated to a single bidder

[5] This is officially justified on the grounds that the trade association has superior information about market conditions and can absorb systematic shocks to the market best by setting an appropriate quantity. The estimates in the following suggests that this assertion has to be taken with a grain of salt.

	Min	Avg	Max
Number of bidders	58	70	82
Share of successful bidders	3/68	65%	76/78
Quota, tons	67.6	311.5	630
Share of quota allocated to single bidder	0.0001	0.014	0.47
Clearing price, CHF/kg	3.21	8.22	14.41
Total revenue, CHF	0.6 mio.	2.763 mio.	7.984 mio.

Table 1: *Summary of Auction Characteristics. The total number of registered bidders is* 123*, the total revenue over the* 39 *auctions from 2008 to 2010 considered is CHF* 107 *mio.*

range from a minimum of 0.0001 to a maximum of 0.47. Table 1 summarizes these auction characteristics.

6.2 Estimating w_i^* and W_i^*

I assume the total number of potential bidders to be the number of registered bidders, that is, we have $N = 123$. I apply Algorithm 1 under the assumption that all bidders are symmetric in a given auction, but that bidders are not necessarily symmetric across the different auctions. I distinguish three scenarios:

(S1) *All auctions are identical.* This scenario presumes that the game faced by the bidders is the same in all auctions. Such might for example come about if the auctioneer can fully accommodate systematic shocks that the bidders are faced in the meat market by appropriately setting the quota. Under this scenario, the algorithm to compute w_i^* and W_i^* samples from the full set of bid-functions available. Analogously, the participation probability is computed by using the full set of auction.

(S2) *Neither of the auctions are identical.* This scenario presumes that the game faced by the bidders changes with every auction. Such might be reasonable to assume if we believe that the auctioneer cannot accommodate for systematic shocks through the quota. Under this scenario, only the bid-functions submitted for the particular auction are considered, and the participation probability is computed by only using the particular auction.

(S3) *Each auction is sufficiently similar to the two preceding and the two following auctions.* This scenario presumes that the game faced by the bidders changes slowly over the course of the auctions. Such might be the case if the shocks that the bidders are faced with have a common component that changes over time and the auctioneer cannot accommodate for these shocks. As a consequence, only bid-functions from auctions lying close by are to be taken for resampling $S(p)$. Analogously, the participation probability is computed by only using auctions lying close by.

Neither of the scenarios (S1)–(S3) is a priori unreasonable, and without a further theoretical or empirical selection criterion it is hard to assess their respective plausibility. The next section discusses in what sense the estimator Θ_t of the fraction of best response violation is a natural candidate for a such selection criterion among the potential scenarios.[6]

[6]Indeed, the set of possible scenarios is infinite. The point with this section is to show that already for these three simple scenarios, the estimated fraction of best response violation differ considerably.

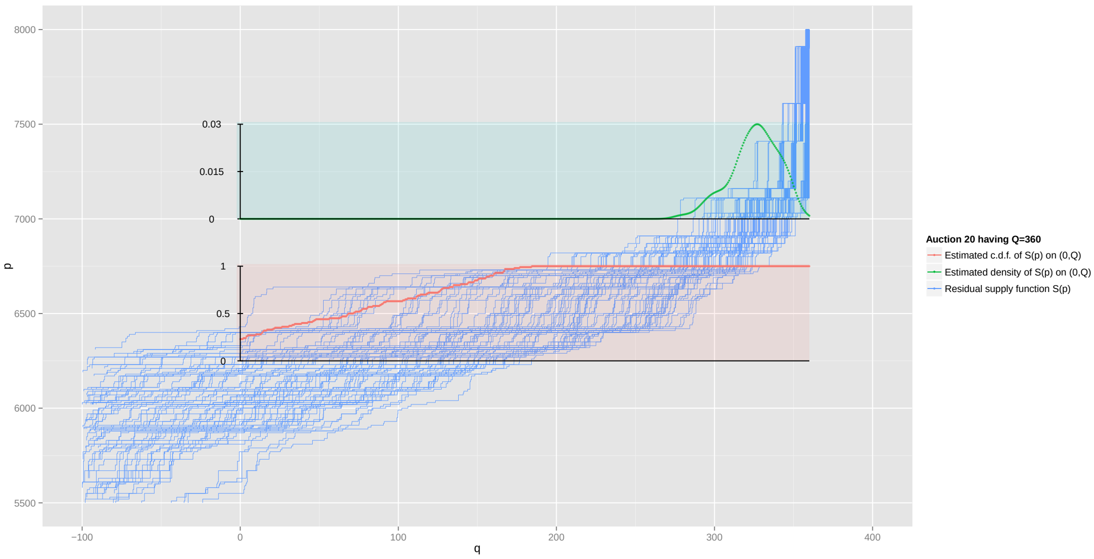

Figure 2: *The figure depicts* 50 *redraws of the residual supply function* $S(p)$ *under scenario (S2) in blue, an estimate of the density of* $S(p)$ *at* $p = 6250$ *in green, and an estimate of the c.d.f. of* $S(p)$ *at* $p = 7000$ *in red.*

The prices in the price-quantity bids are in CHF per ton, and the quantities are in tons. In order to estimate w_i^* and W_i^* for the bidders i in a given auction, I normalize the price-quantity bids of the other auctions from which I draw bid-schedules from to the quota of the auction under consideration. That is, if the quota in the auction t under consideration is Q and that of the auction t' is Q', then, when drawing a bid-schedule from auction t', I multiply each quantity q_i^j in that bid-schedule by Q/Q' and each price p_i^j by Q'/Q.

Figure 2 shows the result of Algorithm 1 by depicting in blue 50 redraws of the residual supply function $S(p)$ under scenario (S2). From these residual supply functions I have computed an estimate of the cumulative distribution function of $S(p)$ for $p = 6250$ in red, and an estimate of the density of $S(p)$ for $p = 7000$ in green, respectively. Whenever we have a bidder i such that there is $j \in \{1, ..., \ell_i\}$ and $p_i^j = 7000$, then we know from Proposition 2 and Lemma 2 that we can use these estimates as estimates for $w_i^*(p_i^j, q)$ whenever $q \in (q_i^{j-1}, q_i^j)$. In a similar manner, the estimate of the cumulative distribution function of $S(p)$ at $p = 6250$ can be taken as estimator of $W_i(p_i^j, q)$ on $q \in (q_i^{j-1}, q_i^j)$ for any bidder i such that there is $j \in \{1, ..., \ell_i\}$ and $p_i^j = 6250$.

6.3 Best response violations

Figure 3 depicts estimates and 95% confidence intervals of Θ_t for $t = 15, ..., 30$ as defined in (24) under the different scenarios (S1)–(S3). The figure suggests that there are systematic and significant differences in the estimates between the scenarios (S1)-(S3): In all but auctions $t = 24$ and $t = 26$ the confidence intervals of the estimates for scenario (S1) and (S2) do not overlap, and the estimates for scenario (S1) are for all these auctions higher than those for scenario (S2). Furthermore, the differences are substantial: If we want to assume that scenario (S1) is valid, then we have to be ready to assume that the fraction of best response violations among the submitted bids is 0.75 or higher with a probability of 90% in eleven out of the 16 auctions under consideration. On the other hand, we see that for eleven out

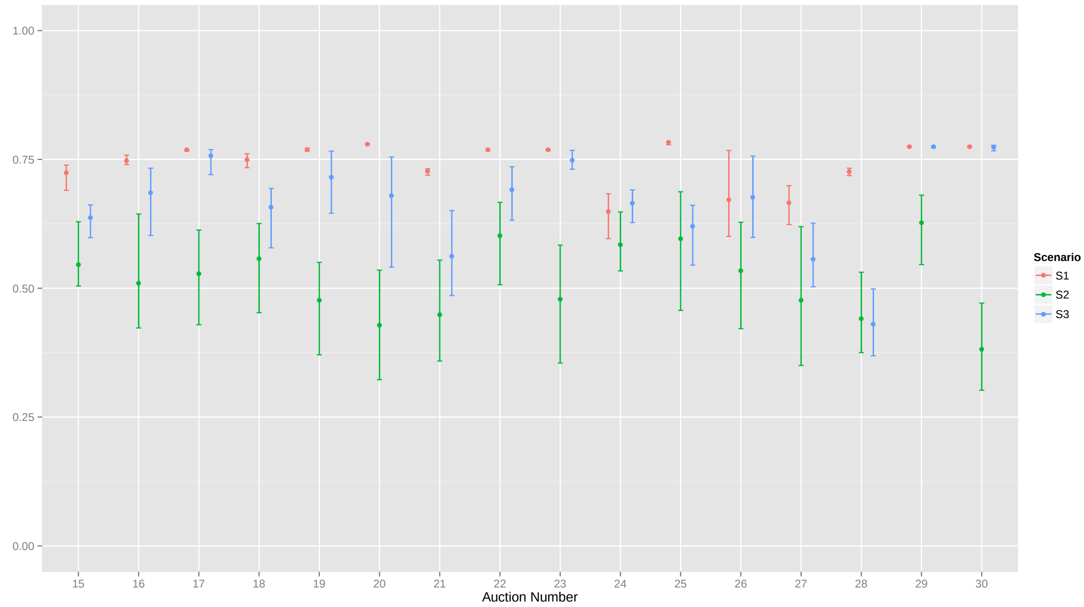

Figure 3: *Estimates of* Θ_t *for auctions* $t = 15,\ldots,30$ *under the different type space scenarios. Estimates and 90% confidence bands computed by using* 100 *bootstrap samples.*

of 16 auctions the fraction of violations lies below 0.5 at a 90% confidence level when we treat all auctions as distinct (i.e. scenario (S2)). For scenario (S3), we get a more mixed picture. Nevertheless, the estimates for scenario (S3) seem to systematically lie between those obtained under scenario (S1) and (S2) – which is not surprising given the fact that scenario (S3) makes less assumptions on the comparability of the different auctions than scenario (S1) but more than scenario (S2).

The findings suggest that the particular type space assumption has a great impact on the plausibility of the estimates. Of course, a thorough check of the type space would also include checking scenarios with subsets of independent and ex ante symmetric bidders that are not symmetric across sets. If so, Algorithm 1 would have to be adapted as discussed above in Section 5.2.

6.4 Computing the Bounds

We now turn to the estimation of the bounds on marginal valuations with Algorithm 2. Algorithm 2 requires to specify the lower and upper bounds $\underline{\omega}^j$ and $\overline{\omega}^j$ on the set of valuation function on which the map g is iterated. A natural candidate for the lower bound is the lower bound obtained on the valuation $v_i(q)$ evaluated at the jth quantity point, i.e. at $q = q_i^j$ as in (23). That is, for bidder i and step $j \in \{1,\ldots\ell_i\}$, I set

$$\underline{\omega}^j = \min\left\{p_i^j, \hat{v}_i^l(q_i^j)\right\}, \tag{29}$$

where I take into account that bidding below one's valuation is dominated (see the proof to Proposition 1). As regards the upper bound $\overline{\omega}^j$ for bidder i and step $j \in \{2,\ldots\ell_i\}$, I resort to

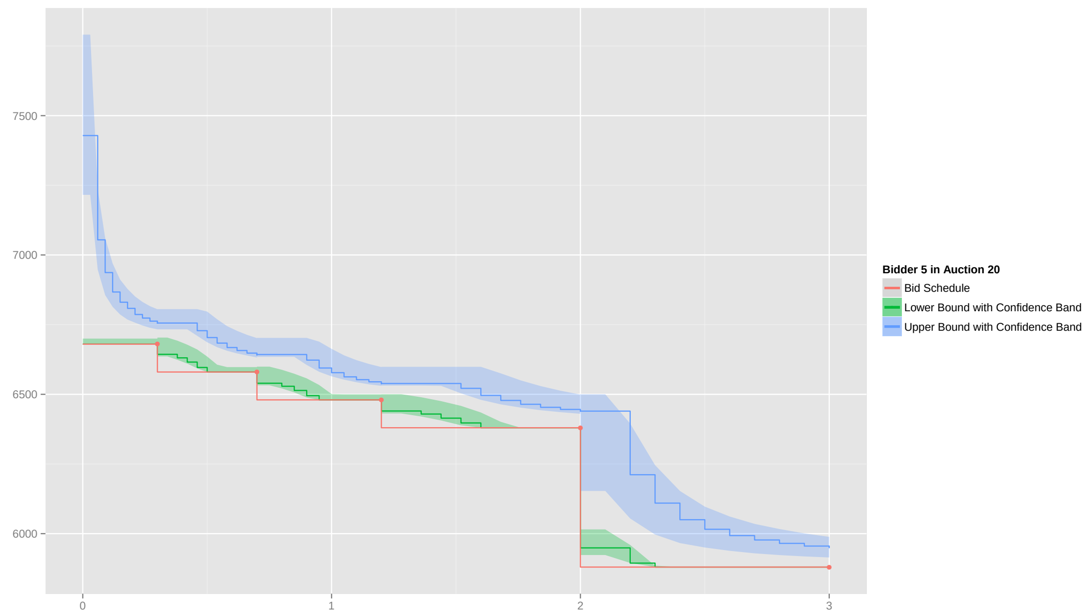

Figure 4: *The result of Algorithm 2 under Scenario (S2) for Bidder* 5 *in Auction* 20*. The chosen error was* $\epsilon = 0.0001$ *and the intervals* $[q_i^j, q_i^{j+1}]$*,* $j \in \{1, \ldots, \ell_i - 1\}$ *were divided into* 10 *segments each. Confidence bands depict pointwise* 95% *confidence intervals for the bounds and were computed by estimating the bounds for* 50 *bootstrap samples of the original data.*

the bound (22) and set[7]

$$\overline{\omega}^j = \hat{v}_i^u(q_i^{j-1}). \tag{30}$$

For the first step $j = 1$ there is no such upper bound available. Because I do not want to assume anything about the upper bound $\overline{v}$ on the type space, I run the algorithm with $\overline{\omega}^j = \infty$ which essentially amounts to running one iteration of g. For any finite $h_q > 0$, this procedure yields a strictly bounded upper bound, while returning the initial condition for the lower bound .

As an example, the resulting bounds from Algorithm 2 for bidder 5 in Auction 20 under scenario (S3) are depicted in Figure 4.[8] Drawn in red is the bid-schedule submitted by the bidder. The blue graph depicts the estimate of the upper bound with pointwise 90% confidence bands in shaded blue. Conversely, the green graph shows the estimate of the lower bound with pointwise 90% confidence bands in shaded green.

Figure 5 depicts the gains in lower bounds on the estimates of the total ex post bidder rent in the auctions $t = 15, \ldots, 30$ (i.e. aggregated ex post valuations of the received quantities minus the payments) when using the Algorithm 2 rather than the estimates obtained when merely using the initial conditions used for the algorithm. The estimates were obtained under scenario (S2) For example, a number of five means that the difference between the estimate when using the algorithm and that when merely using the initial conditions is five percent of the estimate when using the initial conditions. The gains are strictly greater than zero at the

[7]Alternatively, the point estimates of the marginal valuation at the submitted quantities obtained with (21) could have been used. Although this would yield even tighter initial conditions, this only yields sensible results if the estimates are consistent in the sense of Lemma 5. Because I am not convinced that the estimates of W_i^* and w_i^* obtained under either of the scenarios (S1)-(S3) are based on valid assumptions on the type space, I refrain from doing so.

[8]I have chosen (S3) because the resulting bootstrap confidence bands for the bounds tend to be considerably narrower than under (S2), which allows for a better graphical representation.

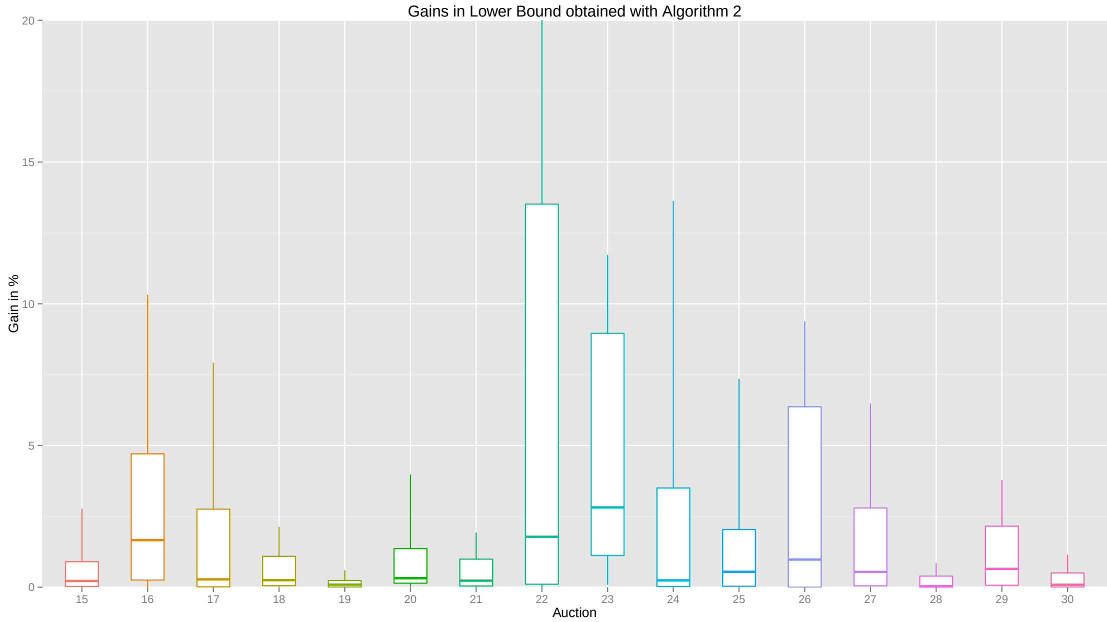

Figure 5: *Boxplots covering* 90% *confidence intervals of gains over estimated lower bound on the estimated ex post rent to bidders in auctions* $t = 15, \ldots, 30$ *estimated for* 80 *bootstrap samples. Upper whisker at third quartile, lower whisker at first, and middle whisker at median.*

90% confidence level for at least five auctions, and furthermore they vary greatly across the auctions. This suggests that any empirical investigation that uses the rent estimates – either aggregate or individual – should take the bounding algorithm into account.

7 Conclusion

This paper has analyzed a discriminatory step function share auction with random participation. It has established equilibrium existence and derived a characterization of the equilibrium bid schedules in terms of the optimality conditions both with respect to the submitted quantities and with respect to the submitted prices. This characterization has yielded a twofold contribution: The paper has, first, presented an estimator for upper and lower bounds on the marginal valuation between the submitted quantity points, and second, it has derived an estimator for the share of submitted price-quantity pairs that violate best response behavior. Both contributions are likely to be important: The former to get narrower bands on rent estimates in discriminatory step function share auctions, the latter to assess the assumptions that the resampling procedure underlying any econometric approach to such share auctions makes on the type space.

A Rationing rules

This appendix studies the pro-rata-on-the-margin rationing rule and the random rationing rule mentioned in the text in some more detail.

A.1 Pro-rata-on-the-margin rationing (Kastl 2011)

Suppose the clearing price p^c is such that there are $m \geq 2$ bidders submitting a bid $p_i^j = p^c$ for some $j \in \{1,...,k\}$ (where the index j possibly differs between the tying bidders). Denote the set of tying bidders by $M \subset \{1,...,n\}$. For every tying bidder $i \in M$, let

$$Q_i = \beta_{b_i}^{-1}(p^c) - \lim_{p \downarrow p^c} \beta_{b_i}^{-1}(p)$$

be the marginal demand at the clearing price p^c. Then, for any bidder $i \in M$, the received quantity is

$$q_i^c = \lim_{p \downarrow p^c} \beta_{b_i}^{-1}(p) + Q_i \cdot \frac{1 - \sum_{j \in N} \lim_{p \downarrow p^c} \beta_{b_j}^{-1}(p)}{\sum_{j \in M} Q_j}.$$

On the other hand, for the non-tying bidders $i \in \{1,...,n\} \setminus M$, we have $q_i^c = \beta_{b_i}^{-1}(p_c)$, and consequently, because q_i^c is deterministic for any profile b and the corresponding clearing price p_c, the distribution $H_i^b(q)$ is given by

$$H_i^b(q) = \begin{cases} 0 & \text{if} \Bigg(\Big(\exists j \in K : p_i^j = p_c \Big) \\ & \qquad \vee \Bigg(q \in \Bigg[0, \lim_{p \downarrow p^c} \beta_{b_i}^{-1}(p) + Q_i \cdot \frac{1 - \sum_{j \in N} \lim_{p \downarrow p^c} \beta_{b_j}^{-1}(p)}{\sum_{j \in M} Q_j} \Bigg) \Bigg) \Bigg) \\ & \qquad \wedge \Big(\Big(\nexists j \in K : p_i^j = p_c \Big) \vee \Big(q \in [0, \beta_{b_i}^{-1}(p_c)) \Big) \Big) \\ 1 & \text{else} \end{cases}.$$

A.2 Random rationing (McAdams 2003)

Suppose the clearing price p^c is such that there are $m \geq 2$ bidders submitting a bid $p_i^j = p^c$ for some $j \in \{1,...,k\}$ (where the index j possibly differs between the tying bidders). Denote the set of tying bidders by $M = \{1,...,m\}$. Let $r = 1 - \sum_{i \in N} \lim_{p \downarrow p^c} \beta_{b_i}^{-1}(p)$ be the excess supply when approaching p^c from above.

For the tying bidders $i \in M$, the allocation procedure is as follows: First, the auctioneer uniformly draws an order $\rho : M \to M$ over the m bidders. Then, the auctioneer starts with the first bidder $\rho(1)$, allocates her

$$q_{\rho(1)}^c = \beta_{b_{\rho(1)}}^{-1}(p^c),$$

and subtracts

$$\beta_{b_{\rho(1)}}^{-1}(p^c) - \lim_{p \downarrow p^c} \beta_{b_{\rho(1)}}^{-1}(p)$$

from r. Then he proceeds to the second bidder $\rho(2)$ in the order, allocates him

$$q_{\rho(2)}^c = \lim_{p \downarrow p^c} \beta_{b_{\rho(2)}}^{-1}(p) + \min\{\beta_{b_{\rho(2)}}^{-1}(p^c) - \lim_{p \downarrow p^c} \beta_{b_{\rho(2)}}^{-1}(p), r\},$$

and again subtracts

$$\beta_{b_{\rho(2)}}^{-1}(p^c) - \lim_{p \downarrow p^c} \beta_{b_{\rho(2)}}^{-1}(p)$$

from r. The procedure goes through the order ρ of bidders, and stops after r has become negative for the first time.

For tying bidder i at clearing price p^c and $q \in [\lim_{p \downarrow p^c} \beta_{b_i}^{-1}(p), \beta_{b_i}^{-1}(p_c))$, let $G_i^b(q)$ denote the distribution of received quantity q_i^c induced by such a procedure. Denoting by $\mathscr{R}$ the set of all orders $\rho : M \to M$ and by $\mathscr{R}_{ij} = \{\rho \in R : \rho(j) = i\}$ the set of all orders such that bidder i is at place j in the order, we have

$$G_i^b(q) = 1 - \sum_{\rho \in \mathscr{R}_{i1}} \frac{1}{m!} - \sum_{j=2}^{m} \sum_{\rho \in \mathscr{R}_{ij}} \mathbb{1}\left\{\sum_{k=1}^{j-1}\left[\beta_{b_{\rho(k)}}^{-1}(p^c) - \lim_{p \downarrow p^c} \beta_{b_{\rho(k)}}^{-1}(p)\right] + q - \lim_{p \downarrow p_c} \beta_{b_i}^{-1}(p)) < r\right\} \frac{1}{m!}.$$

For bidder i and a realized bid-schedule profile $b \in B^n$, the probability $G_i^b(q)$ to win at most $q \in [\lim_{p\downarrow p^c} \beta_{b_i}^{-1}(p), \beta_{b_i}^{-1}(p_c))$ is equal to the probability that the excess supply r is lower than the sum of the marginal demand of those bidders in front of the queue (if there is any) and the marginal demand by bidder i. Because the orders are drawn uniformly, each order $\rho \in \mathbb{R}$ has probability $1/m!$. The discrimination between those orders in which player i is first in line from those orders in which he is not stems from the fact that in the former case the marginal demand by bidder i is always smaller than r.

For the non-tying bidders $i \in \{1,...,n\} \setminus M$, we have $q_i^c = \beta_{b_i}^{-1}(p_c)$. Consequently, under random rationing the distribution $H_i^b(q)$ is for any profile b and corresponding clearing price p_c given by

$$H_i^b(q) = \begin{cases} 0 & \text{if } q \in [0, \lim_{p\downarrow p_c} \beta_{b_i}^{-1}(p)) \\ G_i^b(q) & \text{if } \left(\left(\exists j \in K : p_i^j = p_c\right) \vee \left(q \in \left[\lim_{p\downarrow p^c} \beta_{b_i}^{-1}(p), \beta_{b_i}^{-1}(p_c)\right)\right)\right) \\ 1 & \text{else} \end{cases}.$$

B Proofs of Section 2

Proof of Lemma 1. Let

$$H_i^{(b_i,v_{-i},a_{-i})}(q) = \int\limits_{B^{n-1}} H_i^{(b_i,b_{-i})}(q) \mathrm{d}\mu_1^{a_1}(b_1|v_1)...\mathrm{d}\mu_{i-1}^{a_{i-1}}(b_{i-1}|v_{i-1}) \times$$
$$\mathrm{d}\mu_{i+1}^{a_{i+1}}(b_{i+1}|v_{i+1})...\mathrm{d}\mu_n^{a_n}(b_n|v_n), \quad (31)$$

be the distribution of the quantity q_i^c that bidder i submitting b_i receives when opponents play according to their strategies in μ_{-i}, the opponent type profile is v_{-i}, and the opponent participation profile is a_{-i}. Combining (1) and (31), I write the interim utility $\Pi_i^{a_{-i}}(b_i, v_i, \mu_{-i})$ of player i conditional on the opponent participation profile $a_{-i} \in \{0,1\}^{n-1}$ as

$$\Pi_i^{a_{-i}}(b_i, v_i, \mu_{-i}) = \int\limits_{x \in V^{n-1}} \int_0^1 [V_i(q) - B_i(q)] \mathrm{d}H_i^{(b_i,x,a_{-i})}(q) \mathrm{d}\eta_{-i}(x|v_i), \quad (32)$$

where $V_i(q) = \int_0^q v_i(q)\mathrm{d}q$, and $B_i(q) = \int_0^q \beta_{b_i}(q)\mathrm{d}q$. Because $[V_i(q) - B_i(q)]$ is continuous and $H^{(b_i,x)}(q)$ is monotone, the inner integral of the right-hand side in (32) can be integrated by parts (cf. Apostol, 1974, Theorem 7.6) to yield

$$\int\limits_{x \in V^{n-1}} \left[-\int_0^1 \left[v_i(q) - \beta_{b_i}(q)\right] H_i^{(b_i,x,a_{-i})}(q)\mathrm{d}q \right.$$
$$\left. + [V_i(q) - B_i(q)] H^{(b_i,x,a_{-i})}(q)\Big|_0^1 \right] \mathrm{d}\eta_{-i}(x|v_i). \quad (33)$$

Rearranging, and using the facts that $H_i^{(b_i,x,a_{-i})}(1)=1$, $V_i(0)=0$, $B_i(0)=0$, and

$$\int_{x\in V^{n-1}} \mathrm{d}\eta_{-i}(x|v_i)=1,$$

we can rewrite (33) as

$$-\int_{x\in V^{n-1}}\int_0^1 \left[v_i(q)-\beta_{b_i}(q)\right]H_i^{(b_i,x,a_{-i})}(q)\mathrm{d}q\mathrm{d}\eta_{-i}(x|v_i)+[V_i(1,v_i)-B_i(1)]. \tag{34}$$

Writing the first term in the sum of (34) as

$$-\int_{x\in V^{n-1}}\int_0^1 f(x,q)\mathrm{d}q\mathrm{d}\eta_{-i}(x|v_i), \tag{35}$$

with $f(x,q)=[v_i(q)-\beta_i(q)]H^{(b_i,x,a_{-i})}(q)$ being measurable and bounded on $V^{n-1}\times[0,1]$, the Fubini-Tonelli theorem (cf. Rudin, 1970, Theorem 8.8) can be applied to get that (35) is equal to

$$-\int_0^1 \left[v_i(q)-\beta_{b_i}(q)\right]\int_{x\in V^{n-1}} H^{(b_i,x,a_{-i})}(q)\mathrm{d}\eta_{-i}(x|v_i)\mathrm{d}q. \tag{36}$$

I now define, for $q\in(q_i^{j-1},q_i^j]$,

$$W_i^{j,a_{-i}}(q|b_i,v_i,\mu_{-i})=1-\int_{x\in V^{n-1}} H^{(b_i,x,a_{-i})}(q)\mathrm{d}\eta_{-i}(x|v_i).$$

This allows me to rewrite (36) as

$$-\sum_{j\in K}\int_{q_i^{j-1}}^{q_i^j}\left[v_i(q)-p_i^j\right](1-W_i^{j,a_{-i}}(q|b_i,v_i,\mu_{-i}))\mathrm{d}q. \tag{37}$$

Combining (34) and (37) finally yields

$$\Pi_i^{a_{-i}}(b_i,v_i,\mu_{-i})=\sum_{k\in K}\int_{q_i^{j-1}}^{q_i^j}\left[v_i(q)-p_i^j\right]W_i^{j,a_{-i}}(q|b_i,v_i,\mu_{-i})\mathrm{d}q,$$

from which (2) follows immediately because

$$W_i^j(q|b_i,v_i,\mu_{-i})=\sum_{x\in\{0,1\}^{n-1}} W_i^{j,x}(q|b_i,v_i,\mu_{-i})g(x|a_i=1),$$

and

$$\Pi_i(b_i,v_i,\mu_{-i})=\sum_{x\in\{0,1\}^{n-1}}\Pi_i^{a_{-i}}(b_i,v_i,\mu_{-i})g(x|a_i=1)$$

hold by definition. □

Proof of Proposition 1. For a finite natural h, let $B_h \subset B$ be a discrete action space defined as

$$B_h = \left\{ \left\{p_i^j, q_i^j\right\}_{j=1,\dots,k} \in \left[\left\{0, \frac{\overline{p}}{h}, 2\frac{\overline{p}}{h}, \dots, \overline{p}\right\} \times \left\{0, \frac{1}{h}, \frac{2}{h}, \dots, 1\right\}\right]^k : \right.$$
$$\left. p_i^j \geq p_i^{j+1}, q_i^j \leq q_i^{j+1}, q_i^{k+1} = 1, p_i^{k+1} = 0 \right\}.$$

The strategy in the proof is to first, for any finite natural h, establish existence of an equilibrium μ_h^* in the auction with action space B_h, and then use these equilibria to construct a sequence μ_h^* of equilibria whose limit μ^* is an equilibrium of the game with the unrestricted action space B.

Let $\mathscr{M}_h \subset \mathscr{M}$ be the space of distributional strategies on $V \times B_h$. Define

$$\phi(q; v_i) = \min\left\{p \in \left\{0, \frac{\overline{p}}{h}, 2\frac{\overline{p}}{h}, \dots, \overline{p}\right\} : p \geq v_i(q)\right\}$$

and let $\bar{\mathscr{M}}_h \subset \mathscr{M}_h$ be the set of strategies on $V \times \bar{B}_h$ with

$$\bar{B}_h = \left\{b_i \in B_h : \beta_{b_i}(q) \leq \phi(q; v_i),\ \forall q \in [0,1]\right\}.$$

That is, the strategies in $\bar{\mathscr{M}}_h$ only have those bids in their support such that, as $h \to \infty$, the corresponding step functions lie weakly below the marginal valuation function. I begin with a preliminary lemma that will be used later on.

Lemma 6. *There is $H > 0$ such that for all $h \geq H$, the following holds true: For any bidder $i \in \{1, \dots, n\}$, bidding $b_i \in B_h$ such that $\beta_{b_i}(q) > \phi(q; v_i)$ is weakly dominated. If, for some strategy profile $\mu \in (\mathscr{M}_h)^n$ and share $q \in [q^{j-1}, q^j)$, it holds that $\lim_{x \downarrow q} W_i^j(x|p_i, v_i, \mu_{-i}) > 0$, then bidding $b_i \in B_h$ such that $\beta_{b_i}(q) > \phi(q; v_i)$ is strictly dominated.*

Proof. By contradiction. Suppose $p_i^j > \phi(q; v_i)$ for some $j \in \{1, \dots, k\}$. Take the lowest j for which we find such a p_i^j. Then there exists

$$\bar{q} = \max\left\{q \in [q_i^{j-1}, q_i^j] \cap \{0, 1/h, 2/h, \dots, 1\} : p_i^j \leq \phi(q; v_i)\right\},$$

where I assume the maximum of the empty set above to be q_i^{j-1}. First, suppose $\bar{q} > q_i^{j-1}$ and h large enough, and consider a deviation from (p_i^j, q_i^j) to $(p_i^j + 1/h, \overline{q} - \epsilon_h/h)$, where ϵ_h is given by

$$\epsilon_h = \min\{\delta \in \mathbb{N}_+ : p_i + 1/h \leq v_i(\overline{q} - \delta/h)\}.$$

As $p_i^j + 1/h$ resolves potential ties at p_i^j (which always result in a received quantity q in $(q_i^{j-1}, q_i^j]$ under Assumption (A3) on the tie-breaking rule), the expected avoided loss by such a deviation is bounded from below by the expected loss on the shares in $[\bar{q}, q_i^j]$, that is, by

$$\int_{\bar{q}}^{q_i^j} [p_i^j - v_i(q)] W_i^j(q, |b_i, v_i, \mu_{-i}) \mathrm{d}q. \tag{38}$$

On the other hand, the ex post loss from such a deviation is bounded above by

$$\frac{\bar{q} - q_i^{j-1}}{h} + \int_{\bar{q} - \epsilon_h/h}^{\bar{q}} v_i(q) \mathrm{d}q, \tag{39}$$

where the first term is an upper bound on the loss from bidding $p_i^j + 1/h$ rather than p_i^j and winning at least $\bar{q}$ and the second term is an upper bound on the loss from not winning the shares $q \in [\bar{q} - \epsilon_h/h, \bar{q}]$. Because W_i is left-continuous, we can conclude that, if $\lim_{q \downarrow \bar{q}} W_i^j(q|b_i, v_i, \mu_{-i}) > 0$, then there is $H \in \mathbb{N}_+$ such that for all $h > H$ it holds that $(p_i^j + 1/h, \bar{q} - \epsilon_h/h)$ is a strictly profitable deviation: the lower bound on the gain is independent of h, and, because it follows from the fact that v_i is decreasing that $\lim_{h\to\infty} \epsilon_h/h = 0$, the loss vanishes as h approaches infinity. If, on the other hand, we have $\lim_{q\downarrow\bar{q}} W_i^j(q|b_i, v_i, \mu_{-i}) = 0$, then the bidder does not lose by reducing q_i^j to $\bar{q}$ and leaving p_i^j unchanged, resulting in a weakly profitable deviation. Second, suppose $\bar{q} = q_i^{j-1}$. Then, reducing q_i^j to q_i^{j-1} results in a non-negative net gain (as $p_i^{j+1} \leq p_i^j$), and we also have a profitable deviation which will be strict if $\lim_{q\downarrow\bar{q}} W_i^j(q|b_i, v_i, \mu_{-i}) > 0$.

Going in increasing order through the steps $j = 1, \ldots, k$ at which there is $p_i^j > \phi(q_i^j, v_i)$ and adjusting the respective price-quantity pairs (p_i^j, q_i^j) in the manner described in the last paragraphs yields a profitable deviation for h large enough, and thus the claim. □

Lemma 6 allows to show that an equilibrium μ_h^* in the strategy space $(\bar{\mathscr{M}}_h)^n$ exists.

Lemma 7. *There is $H > 0$ such that for any $h > H$, an equilibrium $\mu_h^* \in (\bar{\mathscr{M}}_h)^n$ exists.*

Proof. I begin by establishing that an equilibrium $\mu_h^* \in (\mathscr{M}_h)^n$ exists. To this end, I rewrite the interim utility such that the ex-ante utility can be expressed as in Milgrom and Weber (1985). I write the interim utility for active bidder i that is conditional on a given participation profile $a_{-i} \in \{0,1\}^{n-1}$ as

$$\Pi_i^{a_{-i}}(b_i, v_i, \mu_{-i}) = \int\limits_{B^{n-1}\times V^{n-1}} u^{a_{-i}}(b_i, b_{-i}, v_i) \mathrm{d}\mu_1(b_1|v_1) \ldots \mathrm{d}\mu_{i-1}(b_{i-1}|v_{i-1}) \times \mathrm{d}\mu_{i+1}(b_{i+1}|v_{i+1}) \ldots \mathrm{d}\mu_n(b_n|v_n) \mathrm{d}\eta_{-i}(v_{-i}|v_i), \quad (40)$$

where $u^{a_{-i}}(b_i, b_{-i}, v_i)$ is the utility form the share auction when only those bids are considered that come from active players in the participation profile a_{-i}. The unconditional interim utility is then given by

$$\Pi_i(b_i, v_i, \mu_{-i}) = \sum_{x \in \{0,1\}^{n-1}} \Pi_i^x(b_i, v_i, \mu_{-i}) g(x|a_i = 1), \quad (41)$$

that can be rewritten to

$$\Pi_i(b_i, v_i, \mu_{-i}) = \int\limits_{B^{n-1}\times V^{mn-1}} \left[\sum_{x\in\{0,1\}^{n-1}} u^{a_{-i}}(b_i, b_{-i}, v_i) g(x|a_i = 1) \right] \times \mathrm{d}\mu_1(b_1|v_1) \ldots \mathrm{d}\mu_{i-1}(b_{i-1}|v_{i-1}) \times \mathrm{d}\mu_{i+1}(b_{i+1}|v_{i+1}) \ldots \mathrm{d}\mu_n(b_n|v_n) \mathrm{d}\eta_{-i}(v_{-i}|v_i). \quad (42)$$

Consequently, we can proceed by defining

$$u_i(b, v_i) = \sum_{x\in\{0,1\}^{n-1}} u^x(b_i, b_{-i}, v_i) g(x|a_i = 1)$$

and

$$U_i(\mu) = \int\limits_{B\times V} \Pi_i(b_i, v_i, \mu_{-i}) \mathrm{d}\mu_i,$$

giving us $\pi_i(\mu)$ in equation (3.1) of Milgrom and Weber (1985). Now, with the assumptions made on the type space V, it follows from the Helly's selection theorem (Rudin, 1964) that V is compact, and thus complete and separable. Because the action space B_h is finite it is compact and the utility u_i is uniformly continuous in $b \in (B_h)^n$. Uniform continuity of u_i in $v \in V^n$ follows because u_i is continuous in v and V^n is compact. Hence, the assumptions of Theorem 1 in Milgrom and Weber (1985) are satisfied and we have existence of an equilibrium that we denote by μ_h^*.

In any equilibrium $\mu_h^* \in (\mathscr{M}_h)^n$ with h large enough it must hold, by Lemma 6, for all bidders $i = 1,...,n$ and for a.e $v_i \in V$, that the bids $b_i \in B_h$ in the support $\mu_{i,h}^*(.|v_i)$ satisfy $\beta_{b_i}(q) \leq \phi(q; v_i)$ whenever $\lim_{x \downarrow q} W_i(x|b_i, v_i, \mu_{-i}) > 0$ holds. So, we need to consider the quantities q at which $\lim_{x \downarrow q} W_i(x|b_i, v_i, \mu_{-i}) = 0$ holds. Observe that, if this holds for some $q \in (q_i^{j-1}, q_i^j]$ we have $\lim_{x \downarrow q'} W_i(x|b_i, v_i, \mu_{-i}) = 0$ for all $q' > q$. Hence, there is a minimal such q that we denote by $\bar{q}$. Assuming $\bar{q} \in (q_i^{j-1}, q_i^j]$, bidder i does not loose utility by setting q_i^j to $\bar{q}$ and p_i^{j+1} to zero. Because bidder i does not win more than $\bar{q}$ in equilibrium μ_h^*, such a change does not affect the winning probabilities of the other players and hence has no effect on their utilities. By above arguments, there is an adapted strategy profile $\mu_h' \in (\bar{\mathscr{M}}_h)^n$ that is also an equilibrium. □

Next, I consider a sequence of equilibria $\mu_h^* \in (\bar{\mathscr{M}}_h)^n$. Because the space $\mathscr{M}$ of probability measures on $B \times V$ is compact in the weak*-topology (Milgrom and Weber, 1985), the sequence $\mu_h^* \in (\bar{\mathscr{M}}_h)^n \subset \mathscr{M}^n$ has a subsequence that converges to some $\mu^* \in \mathscr{M}^n$. In order to show that μ^* is an equilibrium of the step function share auction with the unrestricted action space B it is sufficient, by Remark 3.1 in Reny (1999), to show that the auction for action space B is better reply secure when considering strategies in $\bar{\mathscr{M}} = \lim_{h \to \infty} \bar{\mathscr{M}}_h$, and that for every $\epsilon > 0$ there is $H > 0$ such that for every $h > H$ the profile μ_h^* is an ϵ-equilibrium of the auction with action space B.

Definition 6 (Better-Reply Security, cf. Reny 1999). *Game $G = (\bar{\mathscr{M}}, U_i)_{i \in N}$ is better-reply secure if whenever (μ^*, u^*) is in the closure of the graph $\{(\mu, u) : u = U(\mu)\}$ of its vector payoff function U and μ^* is not a Nash equilibrium, then some player i can secure a payoff strictly above u_i^* at μ^*: There exists some $\tilde{\mu}_i$ such that $U_i(\tilde{\mu}_i, \mu_{-i}) > u_i^*$ for all μ_{-i} in some open neighborhood of μ_{-i}^*.*

Definition 7 (ϵ-equilibrium). *A strategy profile $\mu \in \bar{\mathscr{M}}^n$ is an ϵ-equilibrium of game $G = (\bar{\mathscr{M}}, U_i)_{i \in N}$ if $U_i(\hat{\mu}_i, \mu_{-i}) - U_i(\mu) \leq \epsilon$ for every $\hat{\mu}_i \in \bar{\mathscr{M}}$.*

I start by showing better reply security for which I adapt the argument given in Reny (1999) for the multi-unit auction case. To this end, let

$$\mathscr{M}^p = \left\{ \mu \in \bar{\mathscr{M}} : \forall v_i \in V \ \exists b_i \in B \text{ s.t. } \mu_i(b_i|v_i) = 1 \right\} \tag{43}$$

be the set of strategies that assigns to every valuation $v_i \in V$ a pure strategy $b_i \in B$. Further let

$$\bar{U}(\mu_{-i}) = \sup_{\mu_i \in \mathscr{M}^p} U_i(\mu_i, \mu_{-i}), \tag{44}$$

such that the set

$$B^\epsilon(\mu_{-i}) = \left\{ \mu_i \in \mathscr{M}^p : |U_i(\mu_i, \mu_{-i}) - \bar{U}(\mu_{-i})| \leq \epsilon \right\} \tag{45}$$

describes, for every opponent profile $\mu_{-i} \in \mathscr{M}^{n-1}$, the set of strategies $\mu_i \in \mathscr{M}^p$ that yield utility within $\epsilon > 0$ of the supremum $\bar{U}(\mu_{-i})$. The following observation is needed below.

Lemma 8. *Assume (A2) and fix any $\mu_{-i} \in \mathscr{M}^{n-1}$. Then, for every $\epsilon > 0$ sufficiently small and for any $\mu_i \in B^\epsilon(\mu_{-i})$, $U_i(\mu_i, .)$ is continuous at μ_{-i}.*

Proof. By contradiction. Take $\mu_i \in B^\epsilon(\mu_{-i})$ and suppose $U_i(\mu_i,.)$ is not continuous at μ_{-i}. If $U_i(\mu_i,.)$ is not continuous at μ_{-i} then there is bidder $j \in \{1,...,n\}$ and price p^c such that bidder i and bidder j tie at p^c with positive probability, that is, there are $X, Y \subset V$ with $\eta_i(X), \eta_j(Y) > 0$ such that both bidders submit $p_i^j = p^c$ and $p_j^j = p^c$ (where j might be distinct for the two bidders) with positive probability whenever they have valuation $v_i \in X$ and $v_j \in Y$ respectively. By (A2), there are $X', X'' \subset X$ with $\eta_i(X'), \eta_i(X'') > 0$ where $\forall f \in X'$ and $\forall g \in X''$ it holds that $f(q) > g(q)$, $\forall q \in [0,1]$. Hence, there is a set of bidder i types with strictly positive measure that strictly prefer to avoid the tie, and can do so by raising p_i^j to $p_i^j + \delta$ for any $\delta > 0$ small enough. As the increase is strict, we have a contradiction to the assumption that $\mu_i \in B^\epsilon(\mu_{-i})$ for any $\epsilon > 0$ sufficiently small. □

Next, consider some (μ^*, u^*) that is not an equilibrium and that is in the closure of the graph of the payoff function, i.e. that satisfies $u^* = \lim_{n\to\infty} U(\mu_n)$ for some sequence $\mu_n \to \mu^*$. To show better-reply security, we need to establish that there is some bidder i that can secure a payoff strictly above u_i^* by deviating from μ_i^* even if the other bidders also slightly deviate. I distinguish two cases: (i) $U(.)$ is continuous at μ^* and (ii) $U(.)$ is not continuous at μ^*.

Consider first the case of $U(.)$ being continuous at μ^*. Then there is a bidder i, an $\epsilon > 0$ small enough, and some $\mu_i \in B^\epsilon(\mu_{-i}^*)$ such that $U_i(\mu_i, \mu_{-i}^*) > U(\mu^*) = u^*$. As $U_i(\mu_i,.)$ is continuous at μ_{-i}^* by Lemma 8, we have better reply security.

Second consider the case of $U(.)$ being discontinuous at μ^*. Then there is some $\mu_n \to \mu^*$ such that $u^* = \lim_{n\to\infty} U(\mu_n)$, and there must be at least two bidder i and j tying at some p^c with positive probability, that is, there are $X, Y \subset V$ with $\eta_i(X), \eta_j(Y) > 0$ such that both bidders submit $p_i^j = p^c$ and $p_j^j = p^c$ (where j might be distinct for the two bidders) with positive probability whenever they have valuation $v_i \in X$ and $v_j \in Y$ respectively. By (A2), there are $X', X'' \subset X$ with $\eta_i(X'), \eta_i(X'') > 0$ where $\forall f \in X'$ and $\forall g \in X''$ it holds that $f(q) > g(q)$, $\forall q \in [0,1]$. Hence, there is a set of bidder i types with strictly positive measure that strictly prefer to avoid the tie, and can do so by raising p_i^j to $p_i^j + \delta$ for any $\delta > 0$ small enough. Consider tying bidder i: it follows from Lemma 8 that $\exists \epsilon > 0$ such that, for some n high enough, bidder i can deviate to some $\mu_i \in B^\epsilon(\mu_{-i}^*)$ yielding $U_i(\mu_i, \mu_{-i,n}) > U_i(\mu_n)$ and, furthermore, that

$$\lim_{n\to\infty} U_i(\mu_i, \mu_{-i,n}) = U_i(\mu_i, \mu_{-i}^*) > \lim_{n\to\infty} U_i(\mu_n) = u_i^*, \tag{46}$$

implying $U_i(\mu_i, \hat{\mu}_{-i}) > u_i^*$ for all $\hat{\mu}_{-i}$ in a neighborhood of μ_{-i}^*, which gives us better reply security in this case, too.

It remains to establish that the members μ_h^* of the sequence of equilibria $\mu_h^* \to \mu^*$ are ϵ-equilibria of the game with the unrestricted action space B, where $\epsilon \to 0$ as $h \to \infty$. In order to do so, it is sufficient to show that for every $\epsilon > 0$ there is h high enough such that for every action $b_i \in B$ there is a feasible action $b_i^h \in B_h$ such that the ex-post loss from choosing b_i rather than b_i^h is smaller than ϵ, and that this holds uniformly in the strategies μ_{-i} of the other players (cf. Reny, 2011).

So, fix some finite natural h, some bidder i with v_i and any b_i for which it holds that $\beta_{b_i}(q) \le \phi(q; v_i)$, $\forall q \in [0,1]$. If $b_i \in B_h$ then we are done. So consider $b_i \notin B_h$. Let

$$b_i = \left\{(p_i^1, q_i^1), ..., (p_i^k, q_i^k)\right\}, \tag{47}$$

and define

$$b_{i,h} = \left\{(p_{i,h}^1, q_{i,h}^1), ..., (p_{i,h}^k, q_{i,h}^k)\right\}, \tag{48}$$

with

$$p_{i,h}^j = \min\left\{p \in \left\{0, \frac{\overline{p}}{h}, 2\frac{\overline{p}}{h}, \ldots, \overline{p}\right\} : p \geq p_i^j\right\} \tag{49}$$

$$q_{i,h}^j = \min\left\{q \in \left\{0, \frac{1}{h}, \frac{2}{h}, \ldots, 1\right\} : q \leq q_i^j\right\}, \tag{50}$$

for all $j \in \{1, \ldots, k\}$. Above definitions guarantee that $\beta_{b_{i,h}}(q) \leq \phi(q, v_i)$ holds for all $q \in [0,1]$ and hence that $b_{i,h}$ is a feasible action. The ex-post loss sources from switching from b_i to $b_{i,h}$ are threefold:

1. There might be shares q at which it holds that $\beta_{b_{i,h}}(q) > \beta_{b_i}(q)$ and that are won under b_i, that are also won under b_i^h but at a higher price. The loss from such quantities is bounded above by
$$\sum_{m=1}^{j} (q_i^j - q_i^{j-1}) \frac{\overline{p}}{h}. \tag{51}$$
2. There might be shares q at which it holds that $\beta_{b_{i,h}}(q) > v_i(q) \geq \beta_{b_i}(q)$ and that are not won under b_i, but that are won under b_i^h. The loss from such quantities is bounded above by
$$\sum_{m=1}^{j} (q_i^j - q_i^{j-1}) \frac{\overline{p}}{h}. \tag{52}$$
3. There might be shares q at which it holds that $\beta_{b_{i,h}}(q) < \beta_{b_i}(q)$ and that are won under b_i, but that are won under b_i^h. The loss from such quantities is bounded above by
$$\sum_{m=1}^{j} \int_{q_i^j - 1/h}^{q_i^j} v_i(q) \mathrm{d}q. \tag{53}$$

Observing that all bounds (51)–(53) vanish as $h \to \infty$ independently, and hence uniformly, in the strategies μ_{-i} of the other players, we have that μ_h^* is a sequence of ϵ-equilibria of the auction with the unrestricted action space B where $\epsilon \to 0$ when $h \to \infty$ as required. □

Proof of Proposition 2. To start, observe that from Lemma 8 it follows that ties happen with probability zero in equilibrium. Hence, for a.e. $v_i \in V$ and all bidders i, it holds at the submitted p_i^j and for all $q \in [q_i^{j-1}, q_i^j]$ that

$$W_i^*(p_i^j, q | v_i) = 1 - \sum_{x \in \{0,1\}^{n-1}} F_{S_i(p_i^j)}^{\mu^*, x}(q | v_i) g(x | a_i = 1), \tag{54}$$

where the distribution $F_{S_i(p_i^j)}^{\mu^*, x}(q | v_i)$ is atomless on $[q_i^{j-1}, q_i^j]$ and we have

$$F_{S_i(p_i^j)}^{\mu^*, x}(q | v_i) = 1 - F_{S_i^{-1}(q)}^{\mu^*, x}(p_i^j | v_i)$$

for all $x \in \{0,1\}^{n-1}$ from which it follows that

$$\frac{\partial W_i^*(p_i^j, q | v_i)}{\partial p_i^j} = \sum_{x \in \{0,1\}^{n-1}} f_{S_i(p_i^j)}^{\mu^*, x}(q | v_i) g(x | a_i = 1). \tag{55}$$

for almost every $q \in [q_i^{j-1}, q_i^j]$. Furthermore, it follows from (54) that the equilibrium interim utility of agent i is given by

$$\Pi_i(b_i, v_i, \mu_{-i}^*) = \sum_{k=1}^{j} \int_{q^{j-1}}^{q^j} \left[v_i(q) - p_i^j\right] W_i^*(p_i^j, q|v_i) \mathrm{d}q.$$

Putting for $j = 1, ..., k$ multiplier λ_j on the constraint that $p_i^j \leq p_i^{j-1}$ and multiplier φ_j on the constraint that $q_i^j \geq q_i^{j-1}$, the Lagrangian associated with agent i's optimizing problem is given by

$$L = \sum_{j=1}^{k} \int_{q^{j-1}}^{q^j} \left[v_i(q) - p_i^j\right] W_i^*(p_i^j, q|v_i) \mathrm{d}q - \sum_{k=1}^{j+1} \lambda_j (p_i^j - p_i^{j-1}) - \sum_{k=1}^{j+1} \varphi_j (q_i^{j-1} - q_i^j), \tag{56}$$

where $p_i^{k+1} = q_i^0 = 0$, $p_i^0 = \overline{p}$, and $q_i^{k+1} = 1$. The optimality conditions obtained by setting $\partial L / \partial q_i^j = 0$ for $j = 1, ..., k$ are given by

$$\left[v_i(q_i^j) - p_i^j\right] W_i^*(p_i^j, q_i^j|v_i) - \left[v_i(q_i^j) - p_i^{j+1}\right] W_i^*(p_i^{j+1}, q_i^j|v_i) - (\varphi_{j+1} - \varphi_j) = 0. \tag{57}$$

If we observe $\ell_i = k$ price-quantity points (p_i^j, q_i^j) submitted, with $q_i^j \in (0,1)$, $\forall j \in \{1, ..., k\}$, then the bid-schedule b_i is characterized by the conditions (57) where we have $\varphi_j = 0$ for all $j \in \{1, ..., k\}$. If, however, we observe $\ell_i = k-1$ bid points submitted with $q_i^j \in (0,1)$, $\forall j \in \{1, ..., \ell_i\}$, then $\varphi_j > 0$ holds for some $j \in \{1, ..., k\}$. With $\varphi^j > 0$, the optimality condition (57) for step j is given by

$$\left[v_i(q_i^j) - p_i^j\right] W_i^*(p_i^j, q_i^j|v_i) - \left[v_i(q_i^j) - p_i^{j+1}\right] W_i^*(p_i^{j+1}, q_i^j|v_i) + \varphi_j = 0, \tag{58}$$

and the optimality condition (57) for step $j-1$ is given by

$$\left[v_i(q_i^{j-1}) - p_i^{j-1}\right] W_i^*(q_i^{j-1}, p_i^{j-1}|v_i) - \left[v_i(q_i^{j-1}) - p_i^j\right] W_i^*(q_i^{j-1}, p_i^j|v_i) - \varphi_j = 0. \tag{59}$$

Adding (58) and (59) and taking into account that $q_i^{j-1} = q_i^j$, we arrive at

$$\left[v_i(q_i^{j-1}) - p_i^{j-1}\right] W_i^*(q_i^{j-1}, p_i^{j-1}|v_i) - \left[v_i(q_i^{j-1}) - p_i^{j+1}\right] W_i^*(q_i^{j-1}, p_i^{j+1}|v_i) = 0. \tag{60}$$

All $q_i^{j-1}, p_i^{j-1}, p_i^{j+1}$ are observable, they can be relabeled with indices $j = 1, ..., \ell_i$, such that we have for all $j \in \{1, ..., \ell_i\}$

$$\left[v_i(q_i^j) - p_i^j\right] W_i^*(q_i^j, p_i^j|v_i) - \left[v_i(q_i^j) - p_i^{j+1}\right] W_i^*(q_i^j, p_i^{j+1}|v_i) = 0. \tag{61}$$

As we can repeat this argument for any $\ell_i < k$ observed, characterization (61) holds for any $\ell_i \leq k$. It remains to look at $q_i^{\ell_i} = 1$. Because we have $p_i^{\ell_i+1} = 0$ by definition, the first order condition reads as

$$\left[v_i(q_i^{\ell_i}) - p_i^{\ell_i}\right] W_i^*(q_i^{\ell_i}, p_i^{\ell_i}|v_i) - \varphi_{\ell_i+1} = 0,$$

where $\varphi_{\ell_i+1} = 0$ whenever $q_i^{\ell_i} \in (0,1)$ and $\varphi_{\ell_i+1} \geq 0$ whenever $q_i^{\ell_i} = 1$, yielding the claim with respect to (5).

Because the distribution of $S_i(p_i^j)$ is continuous on $[q_i^{j-1}, q_i^j]$ it follows from (55) that $\partial L/\partial p_i^j = 0$ is equivalent to

$$\int_{q^{j-1}}^{q^j} \left[\left[v_i(q) - p_i^j\right] w_i^*(p_i^j, q|v_i) - W_i^*(p_i^j, q|v_i)\right] \mathrm{d}q - (\lambda_j - \lambda_{j+1}) = 0.$$

If we observe $\ell_i = k$ submitted bid points, then $\lambda_j = 0$, $\forall j \in \{1, ..., k\}$, and we arrive at

$$\int_{q^{j-1}}^{q^j} \left[\left[v_i(q) - p_i^j\right] w_i^*(p_i^j, q|v_i) - W_i^*(p_i^j, q|v_i)\right] \mathrm{d}q = 0.$$

If we observe $\ell_i = k-1$ bid points submitted, then λ_j is non-zero for some $j \in \{1, ..., k\}$. By an analogous argument as in the case of $\varphi_j > 0$ above, we arrive at

$$\int_{q_i^{j-1}}^{q_i^{j+1}} \left[\left[v_i(q) - p_i^j\right] w_i^*(p_i^j, q|v_i) - W_i^*(p_i^j, q|v_i)\right] \mathrm{d}q = 0$$

By appropriately relabeling the observed bid points and repeating the argument in order to cover all cases $\ell_i < k$, we conclude that

$$\int_{q_i^{j-1}}^{q_i^j} \left[\left[v_i(q) - p_i^j\right] w_i^*(p_i^j, q|v_i) - W_i^*(p_i^j, q|v_i)\right] \mathrm{d}q = 0$$

holds for all $j \in \{1, ..., \ell_i\}$, $\forall \ell_i \leq k$. Lastly, we need to look at $p_i^1 = \overline{p}$. If so, the first order condition reads as

$$\int_{0}^{q_i^1} \left[\left[v_i(q) - p_i^1\right] w_i^*(q, p_i^1|v_i) - W_i^*(q, p_i^1|v_i)\right] \mathrm{d}q - \lambda_1 = 0$$

where $\lambda_1 \geq 0$, finally yielding the claim with respect to (6). □

C Proofs of Section 3

Proof of Lemma 3. Let

$$W \equiv \left\{ v \in V_i^j : \underline{\omega}^j \leq v \leq \overline{\omega}^j \right\},$$

let the partial order on W be the point-wise partial order as defined in Definition 4 and let the partial order on $W \times W$ be the order defined in Definition 5. By the order on $W \times W$, the set $W \times W$ being the product lattice of the complete lattice W is again a complete lattice. Further, g_u maps W into W, g_l maps W into W, and both g_u and g_l are order-reversing under the order on W. Hence, by the order on $W \times W$, the function $g : W \times W \to W \times W$ is order preserving. Consequently the set of fixed points $\{v \in W \times W : v = g(v)\}$ is a complete lattice by the Knaster-Tarski-Theorem. By construction of g in (12)–(13) it holds that $\Phi(g) = \{v \in W \times W : v = g(v)\}$, and the statement follows. □

Proof of Proposition 3. It suffices to establish that $(\underline{v}_0^j, \overline{v}_0^j) \in \Phi(g)$. From the fact that the valuations $v_i^j \in V_i^j$ are decreasing it follows that for all $\tilde{q} \in [q_i^{j-1}, q_i^j]$ the functions $v_{\tilde{q}}^u : [q_i^{j-1}, q_i^j] \to \mathbb{R}$ and $v_{\tilde{q}}^l : [q_i^{j-1}, q_i^j] \to \mathbb{R}$ defined as

$$v_{\tilde{q}}^u(q) = \begin{cases} \max\{\overline{v}_0^j(\tilde{q}), \underline{v}_0^j(q)\} & \text{if } q \leq \tilde{q} \\ \underline{v}_0^j(q) & \text{if } q > \tilde{q} \end{cases}$$

$$v_{\tilde{q}}^l(q) = \begin{cases} \overline{v}_0^j(q) & \text{if } q \leq \tilde{q} \\ \min\{\underline{v}_0^j(\tilde{q}), \overline{v}_0^j(q)\} & \text{if } q > \tilde{q} \end{cases}$$

satisfy

$$\int_{q_i^{j-1}}^{q_i^j} \Big[\Big[v_{\tilde{q}}^u(q) - p_i^j\Big] w_i^*(p_i^j, q) - W_i^*(p_i^j, q)\Big] \mathrm{d}q = 0 \tag{62}$$

$$\int_{q_i^{j-1}}^{q_i^j} \Big[\Big[v_{\tilde{q}}^l(q) - p_i^j\Big] w_i^*(p_i^j, q) - W_i^*(p_i^j, q)\Big] \mathrm{d}q = 0. \tag{63}$$

But then, by the construction of g_u and g_l in (12)–(13) and from the fact that the left-hand side of (62) is strictly increasing in $\overline{v}_0^j(\tilde{q})$ whenever $\overline{v}_0^j(\tilde{q}) \geq \underline{v}_0^j(\tilde{q})$ and that the left-hand side of (63) is strictly decreasing in $\underline{v}_0^j(\tilde{q})$ whenever $\overline{v}_0^j(\tilde{q}) \geq \underline{v}_0^j(\tilde{q})$, this implies that $(\underline{v}_0^j, \overline{v}_0^j) \in \Phi(g)$. □

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